Algebraic groups - why, how, what?

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Geometric and topological analogous

Topological group

A topological space which is also a group
$\Rightarrow$ topological $+$ algebraic structure
The group operations (binary operation and inversion) have to be *continuous* with respect to the topology.
Examples: $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{Q}$, $\mathbb{Z}_p$.

Lie group

A smooth (finite dimensional real) manifold which is also a group
$\Rightarrow$ differential geometric $+$ algebraic structure
The group operations have to be *smooth maps*.
Examples: $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{R})$,...
Related: *Complex Lie groups*

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Idea

Combine two structures/approaches in one object, in a compatible way.
In both example above one requires that the group operations are morphisms in the respective category.

Slogan

A (–) group is a group object in the appropriate category.

We want to study the algebro-geometric analogue:

Categorically

An algebraic group is a group object in the category of varieties over a field k.

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Basic examples

Think of an algebraic group G over a field k as a *functor* that associates to a field extension k'/k the set of common solutions over k' of a family of polynomials with coefficients in k.

General linear group

$$\mathrm{GL}_n : k'/k \mapsto \mathrm{GL}_n(k')$$

given by the solutions (t, X), $t \in k'$, $X \in \mathcal{M}_{n\times n}(k')$ of the equation $t \cdot \mathrm{det} X = 1$.

Multiplicative group

$$\mathbf{G}_m : k'/k \mapsto \mathbf{G}_m(k')$$

given by the solutions $(x, y) \in (k')^2$ to the equation $xy = 1$.

Roots of unity

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Building blocks

Every algebraic group can be constructed by extension from algebraic groups of five types:

- finite algebraic groups
- abelian varieties
- semisimple algebraic groups
- algebraic tori
- unipotent groups

Often one restricts attention to *connected, smooth* algebraic groups over e field k.

Definition

An algebraic group is connected if its only finite quotient group is trivial.

Most important classes:

- *affine* algebraic groups
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Affine algebraic groups

These are exactly the (Zariski) closed algebraic subgroups of the matrix groups GL_n.

$\Rightarrow$ *linear* algebraic groups

Important subclass: *reductive* groups:

Definition

A connected affine algebraic group is reductive if it has no connected normal unipotent subgroup other than 1.

Example: GL_n

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Projective algebraic groups

They are automatically abelian.

$$\Rightarrow \textit{abelian varieties}$$

In $\dim = 1$ these are exactly the *elliptic curves*:

Remark

"Elliptic" comes from "elliptic functions", with natural domains Riemann surfaces – an elliptic curve in complex geometry.

- Provides geometric tools to study abelian functions.
- Important in number theory, algebraic geometry, but also in the study of dynamical systems.

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Non-affine and non-projective examples arise naturally!

- X/k proper scheme over a perfect field.

  $(\mathrm{Pic}^0_{X/k})_{red}$: the reduced connected component of its relative Picard scheme

  is in general not proper and not affine.
- V a dvr with fraction field K and residue field k, A/K an abelian variety.

  A_k°: the connected component of the closed fibre of its Néron model

  is an algebraic group over k, in general not proper and not affine.

Question

How far is a smooth connected algebraic group from being affine or projective?

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Chevalley's structure theorem

Every smooth connected algebraic group is "made up" of an abelian variety by a smooth affine algebraic group.

Theorem (Chevalley)

If k is perfect, then every smooth connected algebraic k-group G fits into a unique short exact sequence (to be defined later in the course)

$$1 \to H \to G \to A \to 1$$

where H is linear algebraic and A is an abelian variety.

During the first part of the class, we will develop the language and tools for a (modern) proof this theorem.

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Consequence

An easy consequence is:

Corollary

Any algebraic group G over a field k is necessarily quasi-projective.

A very important consequence is

Néron–Ogg–Shafarevich criterion

Let A be an abelian variety over a local field K and ℓ a prime not dividing the characteristic of the residue field of K. Then A has good reduction if and only if the ℓ-adic Tate module of A is unramified.

Important results

Depending on the time that we have and on the interest of the participants we shall study the following topics.

1. Chevalley's structure theorem
2. Jordan decomposition
3. Unipotent, nilpotent, solvable groups
4. Action of a torus on a smooth projective scheme
5. Reductive groups
6. Torsors,...

Thank you!

Thank you for your attention, now lets get started!