De Jong's alterations

Preliminaries to understand the work of Beilinson
on the p-adic period morphism
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Let X/k be a variety. In general, things are simpler, when X is smooth. If this is not the case, one tries to get them this way. Ideally one would like to resolve the singularities. IN characteristic 0, there have already been results made during the sixties, above all Hironaka's theorem (1964) which says roughly

"Any variety over characteristic 0 can be modified into a nonsingular variety."

In positive characteristic the situation turned out to be more complicated, since only 30 years later, de Jong (1995) obtained a result that conveys the idea that

"Any variety can be altered into a nonsingular variety."

Both statements use certain maps, modifications in the first case and alterations in the second case. Note that alterations are weaker than modifications, thus if one generalises to characteristic $p \neq 0$, one pays the price of allowing more general maps between varieties.

In this note, based on my talk in the Oberseminar on p-adic Hodge theory during the winter term 2014/15, I want to explain some details and give a sketch of the proof of de Jong's theorem. The main references are

- Aise Johan de Jong: *Smoothness, semi-stability and alterations.* [2]
- Dan Abramovich, Frans Oort: *Alterations and resolutions of singularities.* [1]

1 Preliminaries

For the rest of the talk let X/k be a varity over a field k, which means, a separated scheme of finite type over $\operatorname{Spec} k$ which is reduced and irreducible (=integral).

1.1 Operations on varieties

Definition 1.1. Let S be a Notherian scheme. A modification of S is a proper, birational morphism

$$\varphi : S' \to S$$

where S' is integral. The centre of φ is the closed subset of S where φ is not an isomorphism.

Remark 1.2. A modification is flat if and only if it is an isomorphism.

Alterations are more general in that we allow now finite extensions of the function field.

Definition 1.3. Let S be as above. An alteration of S is a dominant and proper morphism

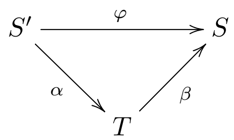
$$\varphi : S' \to S$$

with S' integral such that there is an open subset $\emptyset \neq U \subset S$ with $\varphi^{-1}(U) \to U$ finite (or equivalently, $\dim S' = \dim S$).

Remark 1.4. There is an open "largest" subscheme $\emptyset \neq U \subset S$ such that $\varphi^{-1}(U) \to U$ is finite and flat. The complement of this set U is called the center of φ.

It is clear that the composition o two alterations is again an alteration.

Example 1.5. Every modification is an alteration. MOre precisely, for an alteration $\varphi : S' \to S$ there is a factorisatoin



where $\alpha : S' \to T$ is a modification and $\beta : T \to S$ is finite.

Examples 1.6.

- Blowing up gives naturally a modification.
- Consider the curve $C : x^2 = y^2$ which has a cusp. Then $\mathbb{A}^1 \to C$, $u \mapsto (u^2, u^3)$ is a resolution of C.
- Consider a non-singular variety V, e.g. $\mathbb{C}^2$, with an action of a finite group G on V, e.g. $G = \{$ reflection on origin $\} \cong \mathbb{Z}/2$ on $\mathbb{C}^2$. Then $V \to G\backslash V$ is an alteration, but in general not a modification.

1.2 Normal crossings

To make sense of the term "ordinary nodes" we need the following definition.

Definition 1.7. Let S be a Noetherian scheme, $D \subset S$ a divisor, $D_i \subset D$ for $i \in I$ its irreducible components. Then D is a strict normal crossings divisor if

1. for all $s \in D$ the local ring $\mathscr{O}_{S,s}$ is regular
2. D is a reduced scheme: $D = \bigcup D_i$
3. For all $\emptyset \neq J \subset I$, $D_J = \bigcap_{i \in J} D_i$ is regular of codimension $\#J$ in S.

D is a normal crossings divisor if there is a surjective, étale morphism $\psi : S' \to S$ such that $\psi^{-1}(D)$ is a strict normal crossings divisor in S'.

1.3 Stable curves

Definition 1.8. An S-scheme $f : C \to S$ is a semi-stable curve over S (a family of nodal curves, if it is of finite presentation, proper, flat and all geometric fibres are connected reduced curves with at most ordinary double points as singularities.

Definition 1.9. A semi-stable curve $C \to k$ is split, if

1. All irreducible components are geometrically irreducible and smooth.
2. All singular points are k-rational.

A semi-stable curve $C \to S$ is split if for all $s \in S$, the fibre C_s is split over $\kappa(s)$.

1.4 Semi-stable pairs

Let S be a trait (i.e. $\cong \operatorname{Spec} R$ with a R a DVR).

Definition 1.10. A couple (X, Z), where X is an S-scheme, $Z \hookrightarrow X$ a closed immersion, is a semistable pair if

1. $X \to S$ is strict semistable.
2. Z is a strict normal crossings divisor on X.
3. If the "flat part" $Z_f = \bigcup Z_i$ in irreducible components, for all $\emptyset \neq J \subset I$, the intersection $Z_J = \bigcap_{i \in J} Z_i$ is a disjoint union of strict semistable S-varieties.

In particular the Z_J are flat.

2 Results

We recall Hironaka's theorem, to highlight the differences to de Jong's theorem.

Theorem 2.1 (Hironaka). *Let X/k be a variety, k a field of characteristic 0. Then there exists a sequence of modifications*

$$X_n \to X_{n-1} \to \cdots X_1 \to X_0 = X$$

where $X_i \to X_{i-1}$ is a blowing up with non-singular centre, and the center lies over $\operatorname{sing}(X)$. *In particular, $X_n \to X$ is a resolution of singularities.*

De Jong's theorem says the following.

Theorem 2.2 (de Jong). *Let X/k be a variety. There is a separable alteration*

$$Y \to X$$

such that Y is quasiprojective and regular.

But de Jong proves more [2, Theorem 4.1].

Theorem 2.3. *Let X/k be a variety, $Z \subset X$ a proper closed subset. Then there is an alteration*

$$\varphi_1 : X_1 \to X$$

and an open immersion $j_1 : X_1 \to \overline{X}_1$ such that

1. *The variety $\overline{X}_1$ is a projective variety and a regular scheme.*
2. *The union $j_1(\varphi_1^{-1}(Z)) \cup \overline{X}_1 \backslash j_1(X_1)$ is a strict normal crossing divisor in $\overline{X}_1$.*

If k is a perfect field, then φ_1 can be chosen generically étale.

There are several variants of this theorem, and a relative version might be phrased as follows [2, Theorem 6.5].

Theorem 2.4. *Let S be a trait, X an S-variety. There is a finite morphism of traits $S_1 \to S$, an S_1-variety X_1, an alteration of S-schemes*

$$\varphi_1 : X_1 \to X$$

and an open immersion $j_1 : X_1 \to \overline{X}_1$ of S_1-varieties such that

1. *The scheme $\overline{X}_1$ is a projective S_1-variety with geometrically irreducible generic fibre.*
2. *The pair $\left(\overline{X}_1, (\varphi^{-1}(Z)_{red} \cup \overline{X}_1 \backslash j_1(X_1))\right)$ is a strict semi-stable pair.*

The goal is to give a rough (sketch of the) proof of the two stated theorems.

2.1 Sketch of the proof

Very roughly the idea of the proof is as follows.

Projection. Let X be of dimension d. In order to use induction on dimension, one constructs a morphism

$$f : X \to P,$$

such that P is of dimension $d - 1$ and all the fibers are curves.

Desingularisation of the fibres. Applying an alteration on P changes the morphism f from the previous step such that the fibres have only ordinary nodes as singularities. This step is one of the hardest from a technical point of view as it uses stacks/moduli of curves.

Desingularisation of the base. Again applying an alteration makes P of the map $f : X \to P$ regular. This step uses induction on dimension, as $\dim P < \dim X$.

Desingularisation of the total space. Use explicit (standard) methods of respolutions of singularities.

Remark 2.5. The last step is a "simple exercise" in resolution of singularities. However, in order to arrive there, one pays a huge price, namely one has to use alterations, instead of only modifications.

3 Proof of de Jong's Theorem 4.1

Start with the situation as given in the theorem.

3.1 Preparatory steps

The idea here is to reduce step by step to a situation where we have more restraints on the pair (X, Z) by performing alterations $\varphi : X' \to X$ and then setting $Z' = \varphi^{-1}(Z)$. If one can proof the theorem for the new pair (X', Z') it follows automatically for the old pair (X, Z). Specifically, we will show, that without loss of generality, we can make the assumptions presented in the subsequent part.

(i) k is algebraically closed

Let $X^\star \subset X_{\overline{k}}$ an irreducible component, $Z^\star$ the inverse object of Z. Assume we can solve the problem for this situation, i.e. one can find $\varphi_1^\star : X_1^\star \to X^\star$ and $j_1^\star \to \overline{X}_1^\star$ as desired. Then one wants to deduce such data from this for the original situation. And indeed, in this case, there is a finite field extension $k \subset k_1 \subset \overline{k}$ such that $X^\star$, $X_1^\star$, $\overline{X}_1^\star$, $\varphi_1^\star$ and $j_1^\star$ are defined over k_1. Then there are X', X_1, $\overline{X}_1$, $\varphi_1' : X_1 \to X'$ and j_1 defined over k_1 giving rise to those. Set $\varphi_1 : X_1 \to X$ via

$$X_1 \xrightarrow{\varphi} X' \hookrightarrow X \otimes_{k_1} \to X$$

Now we have a quadruple $(X_1, \overline{X}_1, \varphi_1, j_1)$ as desired. To reiterate, if one finds a quadruple for $\overline{k}$, then this is induced by a quadruple over a finite xtension of k, and finite extensions are allowed for alterations.

(ii) X is quasi-projective

This is a direct consequence of Chow's lemma which says

Theorem 3.1. *There is a modification $\varphi : X' \to X$ such that X' is quasi-projective.*

(iii) X is projective

Let $j : X \hookrightarrow \overline{X}$ an open immersion with $\overline{X}$ projective. Set $\overline{Z} := j(Z) \cup (\overline{X} \backslash X$. Suppose there is a quadruple as in the theorem for $(\overline{X}, \overline{Z})$: $\overline{\varphi}_1 : \overline{X}_1 \to \overline{X}$, $\overline{j}_1 : \overline{X}_1 \to \overline{\overline{X}}_1$ (but $\overline{X}_1$ is already proper thus $\overline{j}_1$ can be chosen to be an isomorphism).

Now set $X_1 := \overline{\varphi}^{-1}(j(X))$, $\varphi_1 = \overline{\varphi}_1\big|_{X_1}$, $j_1 : X_1 \to \overline{X}_1$ an inclusion. Then $j_1(\varphi^{-1}(Z)) \cup \overline{X}_1 \backslash X_1 = \overline{\varphi}_1^{-1}(\overline{Z})$ and φ_1 is an alteration.

(iv) Make Z into a divisor

There is a divisor $D \subset X$ such that Z is the support of D. Blow up X along Z, $\varphi : X' \to X$, which is a modification and $Z' = \varphi^{-1}(Z)$ is the reduction of a divisor. Now enlarge Z to a divisor: $Z \subset Z' \subset X$. Assume the theorem proved for (X, Z'). Then $\varphi^{-1}(Z) \subset X_1 = \overline{X}_1$ has pure codimension 1, and contained in a strict normal crossing divisor $\varphi_1^{-1}(Z')$. Then one may assume that this is a strict normal corssings divisor itself.

(v) X is normal

Normalisation is a modification. Thus we may assume that X is normal to begin with, and this proves to be very useful in one later step. However, later we will perform some alterations, that cancelles the normality. Thus one needs to make sure, that normality is not used after performing this step.

3.2 Producing a projection

Here we want to produce a projection with additional nice properties.

First some general facts about projections. Let $Y \subset \mathbb{P}_k^N$ be a projective variety over $k = \overline{k}$, $p \in \mathbb{P}^N \backslash Y$ a point. Then there is a projection $\mathsf{proj}_p : Y \to \mathbb{P}^{N-1}$ depending on p.

Lemma 3.2. *If* $\dim Y < N-1$ *then there is an open subset* $\emptyset \neq U \hookrightarrow \mathbb{P}^N$ *such that if* $p \notin U$ *then the induced projection* $\mathsf{proj}_p : Y \to \Im(\mathsf{proj}_p)$ *is birational.*

Lemma 3.3. *If* $\dim Y = N-1$*, then there exists* $\emptyset \neq U \hookrightarrow \mathbb{P}^N$ *open, such that, if* $p \in U$ *then* $\mathsf{proj}_p : Y \to \mathbb{P}^{N-1}$ *is generically étale.*

Let's come back to the pair (X, Z).

Lemma 3.4. *Let* (X, Z) *satisfy (i)–(iv). Then there is a modification* $\varphi : X' \to X$ *and a morphism* $f : X' \to \mathbb{P}^{d-1}$ *such that*

1. *There is a finite set of non-singular closed points* $S \subset X_{ns} \backslash Z$ *such that* φ *is a blow-up along* S.
2. f *is equidimensional of relative dimension 1.*
3. *The smooth locus of* f *is dense in all fibers.*
4. *Set* $Z' = \varphi^{-1}(Z)$*. Then the restriction* $f\big|Z'$ *is finite and generically étale (étale on an open subscheme of* $\P^{d-1}$.
5. *If in addition (v) holds, at least one fibre of* f *is smooth.*

Proof. Since $\dim X = d$ and $\dim Z = d-1$, the lemmas tell us that there is a projection $\pi : X \to \mathbb{P}^d$ which is a finite morphism, generically étale and such that the restriction $\pi\big|Z$ is birational onto its image.

Let $B \subset \mathbb{P}^d$ be the locus where π is not étale, $p \notin B$. Then by aboves lemma the projection $\mathsf{proj}_p : \pi(Z) \to \mathbb{P}^{d-1}$ is generically étale. Blow up X along $\pi^{-1}(p)$ to get $\varphi : X' \to X$, where

$$X' = \Big\{(x, \ell) \in X \times \mathbb{P}^{d-1} \quad \big| \quad \pi(x) \in \ell\Big\}$$

This ensures that (1.) is satisfied. Define $f : X' \to \mathbb{P}^{d-1}$ as the projection.

Now we look at the fibres. Let $\ell \in \mathbb{P}^{d-1}$. By construction $f^{-1}(\ell) = \pi^{-1}(L)$ where L is the line corresponding to ℓ. Hence f is equidimensional of at most 1.On the other hand, the line L is defined by $d-1$ equations, consequently, f is of dimension at least 1. Thus (2.).

Since $p \notin B$, $\pi^{-1}(p)$ is contained in the regular locus of X, and every fibre has dense smooth locus. Point (3.) follows.

Furthermore, the fact that π is generically étal and finite implies that $f\big|Z'$ is as well, and this shows (4.).

Now assume (v), i.e. X is normal. This allows us to apply Bertini's theorem (fibres are obtained by intersecting X with linear subspaces).

□

Remark 3.5. The last step in the above proof is in fact the only point, where one needs normality. From now on, we are allowed to perform alterations, that destroy normality.

Lemma 3.6. f *can be assumed to have connected fibres.*

Proof. Let $f : X' \to \mathbb{P}^{d-1}$ as in the previous lemma. There is a factorisation of f, the so called stein factorisation

$$f : X' \to Y' \to \mathbb{P}^{d-1}$$

such that $Y' \to \mathbb{P}^{d-1}$ is finite étale and the fibres of $X' \to Y'$ are connected. Thus we may replace $\mathbb{P}^{d-1}$ by Y'. □

To summarise: we may assume that the pair (X, Z) satisfies (i)–(v) and in addition
(vi) there is a morphism $f : X \to P$ such that

1. its fibres are non-empty, connected equidimensional of dimension 1
2. the smooth locus of f is dense in all fibres
3. the generic finer is smooth
4. the restriction $f\big|Z$ is generically étale and finite

3.3 Enlarge the divisor Z

To rigidify the situation, we enlarge Z, so that it meets every fibre sufficiently.

Lemma 3.7. *Let $X \to P$ as above. Then there exists a divisor $H \subset X$ such that*

1. *$f\big|H : H \to P$ is finite and generically étale*
2. *For every irreducible component C of a fibre of f we have:*

$$\#\,\mathrm{smooth}(X/P) \cap C \cap H \geqslant 3$$

where we count points without multiplicity.

Proof. The proof is very explicit and geometric, using hyperplane sections, and Noetherian induction. See [1, Lemma 4.6] □

3.4 Simplify the fibers

This part of the proof uses the deepest ingredient: moduli of curves (or stacks).

First one applies an alteration on P and replace X and Z by pullbacks via this alteration, such that Z is the union of sections of $X \to P$.

Lemma 3.8. *There is a normal variety P_1 and a separable finite morphism $P_1 \to P$ such that for the strict transform*

$$X_1 := \widetilde{X}_{P_1}$$

which is the essential pullback

$$\begin{array}{ccccc} \overline{P_1 \times_P \eta}^{Zar} & \hookrightarrow & P_1 \times_P X & \longrightarrow & X \\ & & \downarrow & & \downarrow \\ & & P_1 & \longrightarrow & P \end{array}$$

and let Z_1 be the inverse image of Z. Then there is a natural number $n \geqslant 3$ and sections $s_i : P_1 \to X_2$, $i = 1, \dots, n$ such that $Z_1 = \bigcup_{i=1}^n s_i(P_1)$.

Proof. Via induction on the degree of the morphism $Z \to P$. Let $Z_1 \subset Z$ be an irreducible component, $P' = Z_1^\nu$ its normalisation then$P' \to P$ is generically étale. Let $X' = \widetilde{X}_{P'}$ and Z' the inverse image of Z. The morphism $P' \to Z$ gives rise to a section $s : P' \to Z'$, so that we may write $Z' = s(P') \cup Z''$ with

$$\deg(Z'' \to P') = \deg(Z \to P) - 1.$$

Now we perform induction on the degree. □

The next step goes into moduli theory. We may think of the generic fibre of $X \to P$ as smooth curve with a number of points marked on it. Let g be the genus and n be the number of marked points.

Recall the stable extension theorem: Let S be a locally Noetherian integral scheme, $U \subset S$ a dense open, $\mathscr{C} \to U$ with sections $s_i^U : U \to \mathscr{C}$, $i = 1, \dots n$, a stable pointed curve. Then there is an alteration $\varphi : T \to S$, and a stable pointed curve $\mathscr{D} \to T$ with sections $\tau_i : T \to \mathscr{D}$, $i = 1, \dots n$, such that for $U' = \varphi^{-1}(U) \subset T$ we have

$$\mathscr{D}\big|_{U'} \xrightarrow{\varphi} U' \times_U \mathscr{C}$$

such that

$$\varphi^* s_i^U = \tau_i.$$

Hence after an alteration $P_1 \to O$ the generic fibre can be extended to a family of curves

$$\begin{array}{ccc} X_1 & \dashrightarrow & X \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} \\ P_1 & \longrightarrow & P \end{array}$$

such that f_1 has nicer fibres, namely nodal curves.

If we then can resolve P_1, we can resolve X_1, too. The problem is, that $X_1 \dashrightarrow X$ is only a rational map. As we can't pull back along rational maps, we have to ensure that it is actually regular.

Without loss of generality replace P by P_1 and X by a strict transform. We end up with a diagram

$$\begin{array}{ccc} C & \overset{\beta}{\dashrightarrow} & X \\ & \searrow & \downarrow \\ & & P \end{array}$$

and show the so called Three Point Lemma:

Lemma 3.9. *Suppose Z meets the smooth locus of every irreducible component of every fibre in at least three points. Then at least after modification of P, the rational map β extends to a morphism $\beta : C \to X$.*

Proof. The proof of this is quite subtle. It uses flattening of the graph of β, and later a criterion due to Serre. We refer to [1, Section 4.7-4.9]. ☐

As a result, we obtain a diagram of morphisms

$$\begin{array}{ccc} X_1 & \longrightarrow & X \\ \downarrow & & \downarrow \\ P_1 & \longrightarrow & P \end{array}$$

and the argumentation outlined above works.

3.5 Induction on dimension

We take up the situation of the previous section. Let $f : X \to P$ such that $\dim P < \dim X$, f has only nodal curves as fibres, and the divisor Z is "nice" as described previously.

Let $\Sigma \subset P$ be the locus where f is not smooth. By induction there is an alterartion $P_1 \xrightarrow{\phi} P$, such that P_1 is non-singular ans $\phi^{-1}(\Sigma)$ is a strict normal crossings divisor. Replace X by the pullback to P_1 and Z by the union with $f^{-1}(\Sigma)$.

3.6 Blowup a nodal family

We are now in the situation that P and $f : X \to P$ are simplified enough such taht X/P has very simple singularities, i.e. they can be resolved explicitely using standard methods in resolution of singularities. This is left to the inclined reader as an exercise.

4 Proof of de Jong's Theorem 6.5

The strategy of the proof isthe same as for the previous theorem.

Remark 4.1. If $X' \to X$ is an alteration, then X' is an S-variety as well.

Thus as before we can alter X and Z such that

1. X is projective over S.
2. Z is a divisor.

A new feature is that we perform base change with respect to S. This means if

$$S' \to S$$

is a finite morphism of traits, consider the base change

$$X_{S'} = X \times_S S'$$

Then for an irreducible component X', $\varphi : X' \to X$ of S'-varieties, is finite dominant, thus an alteration. If one sets $Z'?\varphi^{-1}(Z)$, then having the statement for $(X', Z')/S'$ implies the statement for $(X, Z)/S$.

With possibly ineparable base changes, assume in addition

3. X_η is geometrically reduced and irreducible.

4. X is normal (via mormalisation as before)

5. The smooth locus of X/S is dense in all fibers

Reduce X to the case with (1)-(5) and furthermore a morphism $\pi : X \to \mathbb{P}^d_S$. As before use this to get $f : X' \to \mathbb{P}^{d-1}_S$ with

- fibers are equidimensional of dimension 1 and non-empty
- The smooth locus of f is smooth in all fibers.

Replacing X by X' we may assume in addition to the previous assumptions

6. There is a projective morphism of S varieties, such that the fibers are equidimensional of dimension 1 and the smooth locus of f is dense in all fibres.

Now we use the Three point Lemma as before to make the fibers "nicer". Then again, one uses induction on the dimension and basic resolution of singularities.

Remark 4.2. This is a very rough idea, many details have to be worked out.

References

[1] Dan Abramovich and Frans Oort. Alteration and resolution of singularities. arXiv:math/9806100v1, 1998.

[2] Aise Johan de Jong. Smoothness, semi-stability and alterations. *Publications Mathématiques de l'IHÉS*, 83:51–93, 1996.