CONJECTURES ON L-FUNCTIONS FOR VARIETIES OVER FUNCTION FIELDS AND THEIR RELATIONS
TALK AT A CONFERENCE IN ARITHMETIC ALGEBRAIC GEOMETRY IN MEMORY OF JAN NEKOVÁŘ

by

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Abstract. — We consider versions for smooth varieties X over finitely generated fields K in positive characteristic p of several conjectures that can be traced back to Tate, and study their interdependence. In particular, let A/K be an abelian variety. Assuming resolutions of singularities in positive characteristic, I will explain how to relate the BSD-rank conjecture for A to the finiteness of the p-primary part of the Tate-Shafarevich group of A using rigid cohomology. Furthermore, I will discuss what is needed for a generalisation. (Joint work with Timo Keller (Groningen) and Yanshuai Qin (Berkeley).)

Contents

Thank you for the invitation, it is an honour to be here at this conference in memory of Jan Nekovář. Today I want to talk about some recent results that we obtained together with Yanshuai Qin and Timo Keller concerning variations for varieties over function fields of conjectures formulated by Tate concerning L-functions, and more precisely their relations. In particular, I want to highlight what role p-adic cohomologies, for example rigid cohomology, play in this context.

0.1. Notation. — I will use the following notation:

$$\begin{aligned} k = \mathbb{F}_q \quad &- \quad \text{a finite field of characteristic } p > 0; \\ W(k) \quad &- \quad \text{the ring of Witt vectors of } k; \\ K_0 \quad &- \quad \text{the fraction field of } W(k). \end{aligned}$$

Key words and phrases. — Rigid cohomology, L-functions, Tate conjectures, functions fields.

0.2. Situation. — Then we consider the following situation:

$$\begin{aligned} S/k \quad &- \quad \text{a smooth projective geometrically irreducible variety;} \\ k(S) \quad &- \quad \text{its function field;} \\ X/k(S) \quad &- \quad \text{a smooth projective variety.} \end{aligned}$$

The conjectures we are interested in concern several invariants associated to such a $X/k(S)$. Often it is not possible to give this definition directly, but one has to use a model.

0.3. Definition. — By spreading out, we can obtain a smooth projective morphism $\rho\colon \mathscr{X} \longrightarrow U$ with generic fibre $X/k(S)$. We call this a smooth model, and for simplicity sometimes write $\mathscr{X}/U$.

1. L-functions

We will in particular be interested in L-functions associated to $X/k(S)$. Already here we use a smooth model:

1.1. Definition. — For such a model, we define the associated n^{th} L-function:

$$\begin{aligned} L_n(\mathscr{X}/U, s) &:= \prod_{u \in |U|} \frac{1}{\det(1 - q^{-\deg(u)s}\mathrm{Frob}_u|_{H^n_{\text{ét}}(\mathscr{X}_{\overline{u}}, \mathbb{Q}_\ell)})} = \prod_{u \in |U|} \frac{1}{\det(1 - q^{-\deg(u)s}\mathrm{Frob}_u|_{(R^n\rho_*\mathbb{Q}_\ell)_{\overline{u}}}} \\ &= \prod_{j=0}^{2\dim(S)} \det(1 - q^{-s}\mathrm{Frob}|_{H^j_{\text{ét},c}(U_{\bar{k}}, R^n\rho_*\mathbb{Q}_\ell)})^{(-1)^{j+1}}, \end{aligned}$$

where $\ell \neq p$ is a prime, Frob_u is the geometric Frobenius of the finite field $k(u)$ (in $\mathrm{Gal}(\overline{k(u)}/k(u)) \simeq \widehat{\mathbb{Z}}$), and Frob is the geometric Frobenius of k (in G_k).

1.2. Remark. — Note that for an abelian variety $A/k(S)$ (where one has a canonical smooth model, the so called Néron model), $L_1(\mathscr{A}/U, s)$ is the classically defined L-function of A, $L(A, s)$. (See also [**6**].)

1.3. Remark. — The last equality is the Grothendieck trace formula, and it shows that they are rational functions.

In general, the L-functions depend on the choice of a model. Nevertheless it is possible to extract information of them which is independent of the model:

***1.4. Proposition* (Tate, Serre).** — *Let $X/k(S)$ be a smooth projective geometrically connected variety and $\mathscr{X}/U$ a smooth projective model.*

(i) For $\Re(s) > \dim(S) + \frac{n}{2}$, the L-function $L_n(\mathscr{X}/U, s)$ converges absolutely (to a holomorphic function).

(ii) The zeros and poles of $L_n(\mathscr{X}/U, s)$ in the strip

$$\dim(S) + \frac{n}{2} - 1 < \Re(s) \leqslant \dim(S) + \frac{n}{2}$$

are independent of the choice of a model $\mathscr{X}/U$.

In the definition of the L-functions, we only use ℓ-adic cohomology for $\ell \neq p$. It doesn't matter which ℓ we choose because of the proper base change theorem for étale cohomology.
If we want to make a similar definition for $\ell = p$, using p-adic cohomology, we have to deal with higher push-forwards in the p-adic world. While it should provide the same L-function, the use of p-adic cohomology will be important for our results.

2. Rigid cohomology for smooth varieties over function fields

In analogy with the lisse $\mathbb{Q}_\ell$-sheaves $R^n\rho_*\mathbb{Q}_\ell$ which appear in the L-functions, we would like to use Berthelot's rigid higher direct image sheaves

$$R^n\rho_{\mathrm{rig}\,,*}\mathscr{O}^\dagger_{\mathscr{X}/K_0}.$$

It is defined in terms of de Rham-cohomology of rigid spaes.

2.1. Remark. — Locally this is defined as follows: assume that ρ has a compactification that lifts smoothly to $W(k)$ in the sense that there is a commutative diagram

$$\begin{array}{ccccc} \mathscr{X} & \hookrightarrow & \overline{\mathscr{X}} & \hookrightarrow & \mathfrak{X} \\ \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \rho} & & \downarrow \\ U & \hookrightarrow & \overline{U} & \overset{\rho}{\hookrightarrow} & \mathfrak{U} \end{array}$$

where the left horizontal maps are open immersions into proper k-varieties and the right horizontal maps are closed immersion into formal $W(k)$-schemes which are smooth in a neighbourhood of $\mathscr{X}$ respectively U. Then we set

$$R^n\rho_{\mathfrak{U},\mathrm{rig}\,,*}\mathscr{O}^\dagger_{\mathscr{X}/K_0} := R^n\rho_*(\Omega^\bullet_{]\overline{\mathscr{X}}[_{\mathfrak{X}}/]\overline{U}[_{\mathfrak{U}}}). \tag{1}$$

As this only depends on $\rho : \mathscr{X} \longrightarrow U$ and $U \hookrightarrow \overline{U} \hookrightarrow \mathfrak{U}$, one can use cohomological descent for the general construction as explained in [**4**].

2.2. Remark. — By Berthelot's conjecture (which is still open) this should have a canonical structure as an overconvergent F-isocrystal. It is however known by Lazda–Ambrosi [**2, 7**] that Ogus' convergent higher direct image sheaves have a unique pre-image under the canonical functor $F\text{-Isoc}^\dagger(U) \longrightarrow F\text{-Isoc}(U)$. For most of the constructions it is enough to use this preimage without being able to identify it precisely.
Moreover, by a result due to Shiho [**11**], we have generic overconvergence of $R^n\rho_{\mathrm{rig}\,,*}\mathscr{O}^\dagger_{\mathscr{X}/K_0}$ up to an alteration. In other words, we can always shrink U and take an alteration to obtain on overconvergent isocrystal. Thus for the sake of this talk, we will assume that the $R^n\rho_{\mathrm{rig}\,,*}\mathscr{O}^\dagger_{\mathscr{X}/K_0}$ are overconvergent F-isocrystals.

2.3. Definition. — We define the rigid cohomology of $X/k(S)$ as

$$H^n_{\mathrm{rig}}(X/k(S), K_0) := \lim_{\mathscr{X}/U} H^0_{\mathrm{rig}}(U, R^n\rho_{\mathrm{rig}\,,*}\mathscr{O}^\dagger_{\mathscr{X}/K_0}).$$

taking the limit over all smooth models of $X/k(S)$.

***2.4. Theorem* (Pal, 2022, [9]).** — *The rigid cohomology $H^*_{\mathrm{rig}}(X/k(S), K_0)$ for smooth projective $k(S)$-varieties is as nice as one could hope for (almost a Weil cohomology).*

— *It is well defined, and functorial in X,*
— *has values in F-isocrystals over k,*
— *has a cup product,*
— *satisfies the Künneth formula*
— *and has a cycle class map $\gamma_X\colon \mathrm{CH}^n(X) \longrightarrow H^{2n}_{\mathrm{rig}}(X/k(S), K_0)$.*

2.5. Definition. — We can also use the overconvergent F-isocrystal $R^n\rho_{\mathrm{rig},*}\mathscr{O}^\dagger_{\mathscr{X}/K_0}$ to define the L-function for a given model $\mathscr{X}/U$ of $X/k(S)$ as

$$L_n(\mathscr{X}/U, s) := \prod_{u\in|U|} \frac{1}{\det(1 - q^{-\deg(u)s}\mathrm{Frob}_u \mid_{(R^n\rho_{\mathrm{rig}\,,*}\mathscr{O}^\dagger_{\mathscr{X}/K_0})_u})},$$

where $\mathrm{Frob}_u = \phi_u^{\deg(u)}$, ϕ_u induced on the fibre at u, as we have an F-isocrystal. That this coincides with the ℓ-adic definition was shown by Pal [**9**].

3. Some conjectures by Tate

I will recall some conjectures that have been formulated by Tate in [**12**]. He asked about the relations between these conjectures and proved some relations between them for smooth projective varieties over finite fields [**12, 13**]. We are now interested in generalising these relations over a function field over k.

3.1. Conjecture $r\mathrm{BSD}\,(X/k(S))$**.** — *Let $X/k(S)$ be a smooth projective geometrically connected variety. The Birch–Swinnerton-Dyer rank conjecture $r\mathrm{BSD}\,(X/k(S))$ predicts the equality of the analytic rank*

$$\mathrm{rk}_{\mathrm{an}}\,(X/k(S)) := \mathrm{ord}_{s=\dim(S)} L_1(\mathscr{X}/U, s),$$

and the algebraic rank

$$\mathrm{rk}_{\mathrm{alg}}\,(X/k(S)) := \mathrm{rk}(\mathrm{Pic}^0(X)),$$

(taking the Prüfer rank of the group $\mathrm{Pic}^0(X) = \mathrm{Pic}^0_{X/k(S)}(k(S))$).

3.2. Remark. — For an abelian variety $A/k(S)$ (assuming that it has a model $\mathscr{A}/U$) which is an abelian scheme, it is not so hard to see the inequality $\mathrm{rk}_{\mathrm{alg}}\,(A/k(S)) \leqslant \mathrm{rk}_{\mathrm{an}}\,(A/k(S))$ for the algebraic and analytic rank. It is possible to use this inequality to prove the $r\mathrm{BSD}\,(A/k(S))$ in some cases using algorithms that provide lower bounds for the algebraic rank.

Next we come to what is usually called **the** Tate conjecture. We only write down the case for divisors, not for arbitrary cycles, as this is the most relevant for us.

3.3. Conjecture $\mathrm{T}^1(X/k(S),\ell)$**.** — *Let $X/k(S)$ be a smooth projective equidimensional variety. The Tate conjecture for divisors $\mathrm{T}^1(X/k(S),\ell)$ says, that either of the cycle class maps*

$$\gamma_X : \mathrm{NS}(X)_{\mathbb{Q}_\ell} := \mathrm{CH}^1_A(X) \otimes \mathbb{Q}_\ell \longrightarrow H^2_{\text{ét}}(X_{k(S)^{sep}}, \mathbb{Q}_\ell(1))^{G_{k(S)}}, \qquad \text{for } \ell \neq p,$$
$$\gamma_X : \mathrm{NS}(S)_{\mathbb{Q}_p} := \mathrm{CH}^1_A(X) \otimes \mathbb{Q}_p \longrightarrow H^2_{\mathrm{rig}}(X/k(S), K_0)^{F=p}, \qquad \text{for } \ell = p,$$

is surjective.

It turns out, that for varying ℓ, the Tate conjectures are all equivalent. The proof of this uses the equivalence to another conjecture which is formulated independently of ℓ:

3.4. Conjecture $\mathrm{T}^1(X/k(S))$**.** — *Let $X/k(S)$ be a smooth projective geometrically connected variety. The conjecture $\mathrm{T}^1(X/k(S))$ states that*

$$\dim_{\mathbb{Q}}(\mathrm{NS}(X)_{\mathbb{Q}}) = -\mathrm{ord}_{s=\dim(S)+1} L_2(\mathscr{X}/U, s),$$

for a model $\mathscr{X}/U \in \mathscr{M}_{X/k(S)}$. As we have seen, the right hand side is independent of the choice of a model.

3.5. Theorem **(Tate, Pal).** — *Let $X/k(S)$ be a smooth projective geometrically connected variety. Then for any prime ℓ*

$$\mathrm{T}^1(X/k(S)) \Longleftrightarrow \mathrm{T}^1(X/k(S),\ell).$$

Thus we just call it the Tate conjecture for divisors.

Another conjecture of Tate that looks somewhat similar to $\mathrm{T}^1(X/k(S))$ is the following:

3.6. Conjecture $\mathrm{T}^1(\mathscr{X})$**.** — *Let $\mathscr{X}$ be a regular scheme of finite type over $\mathbb{Z}$. Then the order of $\zeta(\mathscr{X}, s)$ at the value $s = \dim(\mathscr{X}) - 1$ equals the Euler characteristic $\mathrm{rk}(\mathbb{G}_m(\mathscr{X})) - \mathrm{rk}(\mathrm{Pic}(\mathscr{X}))$.*

3.7. Remark. — Recall that for a regular scheme $\mathscr{X}$ of finite type over $\mathbb{Z}$ the ζ-function is defined as

$$\zeta(\mathscr{X}, s) = \prod_{x \in |\mathscr{X}|} \frac{1}{1 - \operatorname{ord}(\kappa(x))^{-\deg(x)s}}.$$

Let $X/k(S)$ be a smooth projective variety and $(\rho : \mathscr{X} \longrightarrow U, \lambda \colon \mathscr{X}_{k(S)} \xrightarrow{\sim} X) \in \mathscr{M}_{X/k(S)}$. Then we have for the ζ-function of $\mathscr{X}$ is

$$\zeta(\mathscr{X}, s) = Z(\mathscr{X}, q^{-s}) = \prod_{x \in |\mathscr{X}|} \frac{1}{1 - q^{-\deg(x)s}} = \prod_{u \in |U|} \zeta(\mathscr{X}_u, s)$$

and again by the Grothendieck trace formula we have

$$\zeta(\mathscr{X}, s) = \prod_{n=0}^{2\dim(X)} L_n(\mathscr{X}/U, s)^{(-1)^{n+1}}.$$

Tate conjectured the following relation between the conjectures above:

3.8. Conjecture BSD +2. — *Let S be a smooth projective irreducible curve, $X/k(S)$ a smooth projective geometrically connected variety and $\mathscr{X}/U$ a smooth model.*
Then $\mathrm{T}^1(\mathscr{X})$ *is equivalent to* rBSD $(X/k(S)) + \mathrm{T}^1(X/k(S))$.

In fact, this is now a theorem due to Geißer.
We are interested in similar but more general statements. More precisely, we want to give relations between invariants that compute the respective obstructions of the conjectures.

4. Invariants related to these conjectures

There are certain invariants that measure in some sense the obstruction of the conjectures discussed above.

4.1. Notation. — For any abelian group M and prime ℓ, we set

$$\begin{aligned} M[m] &= \{x \in M \mid mx = 0\} \quad \text{for } m \in \mathbb{N} \\ M_{tor} &:= \bigcup_{m \geqslant 1} M[m] \quad \text{(the torsion subgroup)} \\ M[\ell^\infty] = M(\ell) &:= \bigcup_{n \geqslant 1} M[\ell^n] \quad \text{(the } \ell\text{-torsion subgroup)} \\ M(\text{non-}\ell) &:= \bigcup_{m \in \mathbb{N} \ell \nmid m} M[m] \quad \text{(the non-}\ell\text{-torsion subgroup)} \\ T_\ell M &:= \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Q}_\ell/\mathbb{Z}_\ell, M) = \varprojlim_n M[\ell^n] \quad \text{(the } \ell\text{-adic Tate module)} \\ V_\ell M &:= T_\ell M \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \quad \text{(the rational } \ell\text{-adic Tate module)} \end{aligned}$$

The obstruction of the Tate conjecture for divisors can be formulated in terms of the Brauer group.

4.2. Definition. — For a Noetherian scheme $\mathscr{X}$ the cohomological Brauer group is given by

$$\operatorname{Br}(\mathscr{X}) := H^2_{\text{ét}}(\mathscr{X}, \mathscr{O}_{\mathscr{X}}^\times)_{tor}.$$

4.3. Theorem. — *Let $X/k(S)$ be a smooth projective geometrically connected variety. The following statements are equivalent:*

(i) $\operatorname{Br}(X_{k(S)^{sep}})^{G_{k(S)}}(\ell)$ *is finite for some* $\ell \neq p$.

(ii) $\operatorname{Br}(X_{k(S)^{sep}})^{G_{k(S)}}(\text{non-}p)$ *is finite.*

(iii) $\mathrm{T}^1(X/k(S))$ *holds.*

4.4. Remark. — The equivalence between the finiteness of $\mathrm{Br}(X_{k(S)^{sep}})^{G_{k(S)}}(\ell)$ and $\mathrm{T}^1(X/k(S),\ell)$ is well-known. Cadoret–Hui–Tamagawa proved the equivalence between the first two statements. Tate proved the above result when X is defined over a finite field.

Next we come to the obstruction of the BSD-rank conjecture:

4.5. Definition. — Let S be an integral regular noetherian scheme with function field $k(S)$, and A an abelian variety over $k(S)$. The Tate–Shafarevich group of A with respect to S is given by

$$Ш_S(A) := \ker\left(H^1_{\text{ét}}(k(S),A) \to \prod_{s\in S^1} H^1_{\text{ét}}(k(S)^{\text{sh}}_s, A)\right)$$

where S^1 denotes the set of codimension-1 points of S and $k(S)^{\text{sh}}_s$ the fraction field of a strict Henselisation of the local ring $\mathscr{O}_{S,s}$.

***4.6. Theorem* (Keller, Qin).** — *Let $A/k(S)$ be an abelian variety. The following statements are equivalent:*

(i) $Ш_S(A)(\ell)$ *is finite for some* $\ell \neq p$.

(ii) $Ш_S(A)(\textit{non-}p)$ *is finite.*

(iii) $r\mathrm{BSD}\,(A/k(S))$ *holds.*

The above was shown by Keller for good reduction and then generalised by Qin.

4.7. Remark. — Note that in the above theorems the obstruction depends on a prime $\ell \neq p$. We are interested in obtaining similar statements for $\ell = p$.

Now it is possible to related the two conjectures in terms of their ℓ-adic obstructions as follows:

***4.8. Theorem* (Qin).** — *Let $X/k(S)$ be a smooth projective geometrically connected variety and $\mathscr{X}/U$ a smooth model. Assume that* $\mathrm{Br}(S)$ *is finite. (In particular the* $\mathrm{Br}(S)(\ell)$ *is finite and this is equivalent to* $V_\ell\mathrm{Br}(S) = 0$.*) For any $\ell \neq p$, there is an exact sequence*

$$0 \to V_\ell Ш_S(\mathrm{Pic}^0_{X/k(S),red}) \to V_\ell\mathrm{Br}(\mathscr{X}) \to V_\ell\mathrm{Br}(X_{k(S)^{sep}})^{G_{k(S)}} \to 0.$$

4.9. Remark. — Note that the vanishing of $V_\ell Ш_S(A)(\ell)$ is weaker than the statement that $Ш_S(A)(\ell)$ is finite. However, for $\ell \neq p$, it follows from Kummer theory and finiteness statements, that they are equivalent.

Of course it is natural to ask about the p-adic obstructions of the above conjectures and their relation. This is the topic of the last part.

5. The p-adic case

I will assume now that $X/k(S)$ has everywhere good reduction. Under the hypothesis of resolution of singularities, one can treat a slightly more general case, but this will only add some technical arguments and not give more insight in the general case. So for the sake of this talk, I will stick to good reduction. Thus the statement that we want can be formulated as follows:

***5.1. Theorem* (E.–Keller–Qin).** — *Let $X/k(S)$ be a smooth projective geometrically connected variety, and assume that it has a smooth model $\rho : \mathscr{X} \to S$. Assume that* $\mathrm{Br}(S)$ *is finite. (Or* $V_p\mathrm{Br}(S) = 0$, *or* $\mathrm{Br}(S)(p)$ *is finite.) There is a short exact sequence*

$$0 \to V_p Ш_S(\mathrm{Pic}^0_{X/k(S),red}) \to V_p\mathrm{Br}(\mathscr{X}) \to \mathrm{Coker}(\gamma_X) \to 0.$$

with the cycle class map $\mathrm{NS}(S)_{\mathbb{Q}_p} \xrightarrow{\gamma_X} H^2_{\mathrm{rig}}(X/k(S),K_0)^{F=p}$ *(hence this cokernel is the obstruction of the Tate conjecture* $\mathrm{T}^1(X/k(S),p)$ *– which for the moment we don't identify precisely).*

Sketch of proof. — Consider the flat and rigid Leray spectral sequences

$$E_2^{i,j} = H^i_{fppf}(S, R^j\rho_{fppf,*}\mathscr{O}^\times_{\mathscr{X}}) \Longrightarrow H^{i+j}_{fppf}(\mathscr{X},\mathscr{O}^\times_{\mathscr{X}}),$$
$$E_2^{i,j} = H^i_{\mathrm{rig}}(S, R^j\rho_{\mathrm{rig},*}\mathscr{O}^\dagger_{\mathscr{X}/K_0}) \Longrightarrow H^{i+j}_{\mathrm{rig}}(\mathscr{X}/k,K_0).$$

They degenerate at the E_2-page by variants of Deligne's splitting criterion. (For crystalline cohomology this was proved by Morrow [**8**], and in this situation there is a comparison to rigid cohomology.) Thus we obtain short exact sequences fitting into a commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Pic}(S)_{\mathbb{Q}_p} & \longrightarrow & \mathrm{Ker}\big(\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}_p} \to \mathrm{NS}(X)_{\mathbb{Q}_p}\big) & \longrightarrow & \mathrm{Pic}^0(X)_{\mathbb{Q}_p} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \gamma_S} & & \downarrow{\scriptstyle \gamma_{\mathscr{X},X}} & & \downarrow{\scriptstyle \gamma^0_X} & & \\
0 & \longrightarrow & H^2_{\mathrm{rig}}(S/k,K_0)^{F=p} & \longrightarrow & \mathrm{Ker}\big(H^2_{\mathrm{rig}}(\mathscr{X}/k,K_0) \to H^2_{\mathrm{rig}}(X/k(S),K_0)\big)^{F=p} & \longrightarrow & H^1_{\mathrm{rig}}(S,R^1\rho_{\mathrm{rig},*}\mathscr{O}^\dagger_{\mathscr{X}/K_0})^{F=p} & \longrightarrow & 0
\end{array}$$

Here we used that taking $(-)^{F=p}$ on the second sequence is exact by [**5**, II.Lem. 5.6]. Now we can identify the cokernel of γ_S as $V_p\mathrm{Br}(S)$. Moreover, it turns out, that γ^0_X is injective, and has cokernel $V_p\text{Ш}_S(\mathrm{Pic}^0_{X/k(S),red})$. The proof for this generalises some arguments of Bauer in [**3**], where he treats the case when S is a curve. Thus we get a diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
0 & \longrightarrow & \mathrm{Pic}(S)_{\mathbb{Q}_p} & \longrightarrow & \mathrm{Ker}\big(\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}_p} \to \mathrm{NS}(X)_{\mathbb{Q}_p}\big) & \longrightarrow & \mathrm{Pic}^0(X)_{\mathbb{Q}_p} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \gamma_S} & & \downarrow{\scriptstyle \gamma_{\mathscr{X},X}} & & \downarrow{\scriptstyle \gamma^0_X} & & \\
0 & \longrightarrow & H^2_{\mathrm{rig}}(S/k,K_0)^{F=p} & \longrightarrow & \mathrm{Ker}\big(H^2_{\mathrm{rig}}(\mathscr{X}/k,K_0) \to H^2_{\mathrm{rig}}(X/k(S),K_0)\big)^{F=p} & \longrightarrow & H^1_{\mathrm{rig}}(S,R^1\rho_{\mathrm{rig},*}\mathscr{O}^\dagger_{\mathscr{X}/K_0})^{F=p} & \longrightarrow & 0\\
 & & \downarrow & & & & \downarrow & & \\
 & & V_p\mathrm{Br}(S) & & & & V_p\text{Ш}_S(\mathrm{Pic}^0_{X/k(S),red}) & &
\end{array}$$

Applying the snake lemma to it, we obtain an exact sequence

$$0 \longrightarrow V_p\mathrm{Br}(S) \longrightarrow \mathrm{Coker}(\gamma_{\mathscr{X},X}) \longrightarrow V_p\text{Ш}_S(\mathrm{Pic}^0_{X/k(S),red}) \longrightarrow 0.$$

But $\mathrm{Coker}(\gamma_{\mathscr{X},X})$ can be identified with $\mathrm{Ker}\big(V_p\mathrm{Br}(\mathscr{X}) \to \mathrm{Coker}(\gamma_X)\big)$. Indeed, looking at the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}\big(\mathrm{Pic}(\mathscr{X})_{\mathbb{Q}_p} \to \mathrm{NS}(X)_{\mathbb{Q}_p}\big) & \longrightarrow & \mathrm{Pic}(\mathscr{X})_{\mathbb{Q}_p} & \longrightarrow & \mathrm{NS}(X)_{\mathbb{Q}_p} & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle \gamma_{\mathscr{X},X}} & & \downarrow{\scriptstyle \gamma_{\mathscr{X}}} & & \downarrow{\scriptstyle \gamma_X} & & \\
0 & \longrightarrow & \mathrm{Ker}\big(H^2_{\mathrm{rig}}(\mathscr{X}/k,K_0) \to H^2_{\mathrm{rig}}(X/k(S),K_0)\big)^{F=p} & \longrightarrow & H^2_{\mathrm{rig}}(\mathscr{X}/k,K_0)^{F=p} & \longrightarrow & H^2_{\mathrm{rig}}(X/k(S),K_0)^{F=p} & \longrightarrow & 0
\end{array}$$

where again γ_X is injective, and the cokernels are given by

$$\begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & \operatorname{Ker}\big(\operatorname{Pic}(\mathscr{X})_{\mathbb{Q}_p} \to \operatorname{NS}(X)_{\mathbb{Q}_p}\big) & \longrightarrow & \operatorname{Pic}(\mathscr{X})_{\mathbb{Q}_p} & \longrightarrow & \operatorname{NS}(X)_{\mathbb{Q}_p} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \gamma_{\mathscr{X},X}} & & \downarrow{\scriptstyle \gamma_{\mathscr{X}}} & & \downarrow{\scriptstyle \gamma_X} & & \\ 0 & \longrightarrow & \operatorname{Ker}\big(H^2_{\mathrm{rig}}(\mathscr{X}/k, K_0) \to H^2_{\mathrm{rig}}(X/k(S), K_0)\big)^{F=p} & \longrightarrow & H^2_{\mathrm{rig}}(\mathscr{X}/k, K_0)^{F=p} & \longrightarrow & H^2_{\mathrm{rig}}(X/k(S), K_0)^{F=p} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & \operatorname{Coker}(\gamma_{\mathscr{X},X}) & & V_p\operatorname{Br}(\mathscr{X}) & & \operatorname{Coker}(\gamma_X) & & \end{array}$$

and we can again apply the snake lemma to obtain a short exact sequence

$$0 \longrightarrow \operatorname{Coker}(\gamma_{\mathscr{X},X}) \longrightarrow V_p\operatorname{Br}(\mathscr{X}) \longrightarrow \operatorname{Coker}(\gamma_X) \longrightarrow 0.$$

Thus we can substitute as follows

$$0 \longrightarrow V_p\operatorname{Br}(S) \longrightarrow \operatorname{Ker}\big(V_p\operatorname{Br}(\mathscr{X}) \to \operatorname{Coker}(\gamma_X)\big) \longrightarrow V_p Ш_S(\operatorname{Pic}^0_{X/k(S),red}) \longrightarrow 0.$$

As we assumed $V_p\operatorname{Br}(S) = 0$, there is an issomorphism

$$\operatorname{Ker}\big(V_p\operatorname{Br}(\mathscr{X}) \to \operatorname{Coker}(\gamma_X)\big) \cong V_p Ш_S(\operatorname{Pic}^0_{X/k(S),red})$$

which gives the claimed exact sequence. □

This allows us to show the following:

***5.2. Corollary* (E.–Keller–Qin).** — *Let $A/k(S)$ be an abelian variety. Assume that it has a smooth model $\mathscr{A}/S$, and that* $\operatorname{Br}(S)$ *is finite. The following statements are equivalent:*

- *(i)* $Ш_S(A)(p)$ *is of finite exponent.*
- *(ii)* $r\mathrm{BSD}\,(A/k(S))$ *holds.*

Sketch of proof. — For abelian varieties, the Tate conjecture for divisors is known, thus we have

$$V_\ell\operatorname{Br}(A_{k(S)^{sep}})^{G_{k(S)}} = 0, \text{ for } \ell \neq p$$
$$\operatorname{Coker}(\gamma_A) = 0, \text{ for } \ell = p$$

Thus by the above short exact sequences for $\ell \neq p$ and $\ell = p$, we have isomorphisms

$$V_\ell Ш_S(A) \cong V_\ell\operatorname{Br}(\mathscr{A}), \text{ for } \ell \neq p,$$
$$V_p Ш_S(A) \cong V_p\operatorname{Br}(\mathscr{A}).$$

As $V_\ell\operatorname{Br}(\mathscr{A})$ is the obstruction for the Tate conjecture for divisors $\mathrm{T}^1(\mathscr{A}/k)$ for any ℓ, and we know that all Tate conjectures $\mathrm{T}^1(\mathscr{A}, \ell)$ are equivalent, the p-adic case follows simply from the ℓ-adic statement. □

5.3. Remark. — In the above theorem, we had rather strong assumptions. While it is true, that we can get rid of some of them by using resolution of singularities in positive characteristic, this is not ideal. However, in the proof we didn't use any properties of $\operatorname{Coker}(\gamma_X)$, except, that it is the obstruction of $T^1(X/k(S), p)$.

For the case of an abelian variety as in the theorem, the Tate conjecture is of course known. In this case, d'Addezio showed in [**1**] that

$$V_p\operatorname{Br}(A_{k(S)^{sep}})^{G_{k(S)}} = 0.$$

In order to generalise the theorem, and ultimately the corollary, it would be helpful to identify the obstruction of $T^1(X/k(S), p)$ for general (smooth, projective) X. This is work in progress.

5.4. Strategy. — Let again $X/k(S)$ be proper and smooth and let $\rho : \mathscr{X} \to U$ be a smooth model. Recall the $T^1(X/k(S), p)$ predicts the surjectivity of the cycle class map

$$\mathrm{NS}(X)_{\mathbb{Q}_p} \longrightarrow H^2_{\mathrm{rig}}(X/k(S), K_0)^{F=p}.$$

By Kummer theory, there is a natural short exact sequence

$$0 \longrightarrow \mathrm{NS}(X) \otimes \mathbb{Q}_p/\mathbb{Z}_p \longrightarrow H^0_{fppf}(\mathrm{Spec}(k(S)), R^2\rho_*\mu_{p^\infty}) \longrightarrow \mathrm{Br}(X_{k(S)^{sep}})^{G_{k(S)}}(p) \longrightarrow 0.$$

Applying V_p to this sequence, we have

$$V_p H^0_{fppf}(\mathrm{Spec}(k(S)), R^2\rho_*\mu_{p^\infty}) \cong \mathrm{Hom}\,(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, R^2\rho_*\mu_{p^\infty})_{\mathbb{Q}_p}.$$

Thus it now remains to show that

$$\mathrm{Hom}\,(\underline{\mathbb{Q}_p/\mathbb{Z}_p}, R^2\rho_*\mu_{p^\infty})_{\mathbb{Q}_p} \cong H^2_{\mathrm{rig}}(U, R^2\rho_{\mathrm{rig}\,,*}\mathscr{O}^\dagger_{\mathscr{X}/K_0})^{F=p} = H^2_{\mathrm{rig}}(X/k(S), K_0)^{F=p}.$$

We have

$$H^2_{\mathrm{rig}}(U, R^2\rho_{\mathrm{rig}\,,*}\mathscr{O}^\dagger_{\mathscr{X}/K_0})^{F=p} \cong \mathrm{Hom}_{\,F\text{-Isoc}^\dagger}(\mathscr{O}^\dagger_{U/K_0}(1), R^2\rho_{\mathrm{rig}\,,*}\mathscr{O}^\dagger_{\mathscr{X}/K_0}).$$

Moreover, by Kedlaya's full faithfullness theorem, we can identify the latter with

$$\mathrm{Hom}_{\,F\text{-Isoc}}(\mathscr{O}_{U/K_0}(1), R^2\rho_{\mathrm{cris},*}\mathscr{O}_{\mathscr{X}/K_0}) = \mathrm{Hom}_{\,F\text{-Isoc}}(\mathscr{O}_{U/K_0}(1), (R^2\rho_{\mathrm{cris},*}\mathscr{O}_{\mathscr{X}/K_0})_{[0,1]})$$

Now we can use the two facts

— $(R^2\rho_{\mathrm{cris},*}\mathscr{O}_{\mathscr{X}/K_0})_{[0,1]}$ is isogenous to a Dieudonné crystal.

— $R^2\rho_*\mu_{p^\infty}$ up to groups of finite exponent is isomorphic to a p-divisible group over $\mathrm{Spec}(k(S))$.

Applying the Dieudonné functor, we have

$$\mathbb{D}(\underline{\mathbb{Q}_p/\mathbb{Z}_p}) \cong \mathscr{O}_U(1)$$

Thus it suffices now to show that

$$\mathbb{D}(R^2\rho_*\mu_{p^\infty}) \cong (R^2\rho_{\mathrm{cris},*}\mathscr{O}_{\mathscr{X}/K_0})_{[0,1]}$$

as F-isocrystals. This is not known at the moment, but there is strong evidence that this is true. For example, Oda proved

$$\mathbb{D}(R^1\rho_*\mu_{p^\infty}) \cong (R^1\rho_{\mathrm{cris},*}\mathscr{O}_{\mathscr{X}/K_0})_{[0,1]}$$

End

Thank you very much for your attention!

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Version of November 16, 2023

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