RIGID ANALYTIC RECONSTRUCTION OF HYODO–KATO THEORY ALGEBRAIC GEOMETRY SEMINAR AT THE UNIVERSITY OF WARSAW, 2023/24

by

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Abstract. — These are notes for a lecture series in the Algebraic Geometry Seminar at the University of Warsaw during winter term 2023/24. In the first part, I give an introduction to p-adic Hodge theory, which serves as a motivation for the second part. In the second part I explain joint results with Kazuki Yamada (University of Tokyo) on a rigid analytic approach to Hyodo–Kato theory.

Contents

Thank you very much for the oportunity to speak here at the Algeberaic Geometry Seminar of the University of Warsaw. And thank you in particular for the warm welcome to the city. This talk will have two parts. In the first part, I will tell you a little bit about the history of p-adic comparison theorems. This should hopefully provide some motivation for the second part, in which I will explain you a new approach to Hyodo–Kato theory due to Kazuki Yamada and myself based on rigid analytic methods.

Key words and phrases. — Rigid cohomology, log geometry, comparison morphisms.

PART I. CLASSICAL AND P-ADIC COMPARISON THEOREMS

I first want to recall some classical results, which also relate nicely to our Hodge-seminar, and after that go into the p-adic direction.

1. The theorem of de Rham

We will start with the classical situation over complex numbers.

1.1. Notation. — Let $Z/\mathbb{Q}$ be a smooth projective algebraic variety. Consider

$$H_*(Z(\mathbb{C}),\mathbb{C}) \quad - \quad \text{the singular homology of the topological space } Z(\mathbb{C}),$$
$$H^*_{\mathrm{dR}}(Z_\mathbb{C}) := H^*(Z_\mathbb{C},\Omega^\bullet_{Z_\mathbb{C}/\mathbb{C}}) \quad - \quad \text{the algebraic de Rham cohomology of } Z_\mathbb{C}.$$

Now de Rham's theorem tells us that integrating along cycles provides a non-degenerate pairing.

***1.2. Theorem* (de Rham).** — *For any $n \geqslant 0$ the pairing*

$$H^n_{\mathrm{dR}}(Z_\mathbb{C}) \times H_n(Z(\mathbb{C}),\mathbb{C}) \to \mathbb{C},$$
$$(\omega,\gamma) \mapsto \int_\gamma \omega$$

is non-degenerate.

1.3. Remark. — It is possible to extend this to all algebraic varieties by resolution of singularities and appropriately adapting de Rham cohomology and singular cohomology.

Dually, one obtains a comparison isomorphism of $\mathbb{C}$-vector spaces:

1.4. Corollary. — *There is a $\mathbb{C}$-linear comparison isomorphism*

$$H^n_{\mathrm{dR}}(Z_\mathbb{C}) \cong H^n_B(X(\mathbb{C}),\mathbb{C}),$$

where $H^n_B(X(\mathbb{C}),\mathbb{C})$ is singular cohomology, also called Betti cohomology, dual to singular homology by Poincaré duality $\mathrm{Hom}_\mathbb{C}(H_{2d-n}(X(\mathbb{C}),\mathbb{C})\mathbb{C}) \cong H^n(X(\mathbb{C}),\mathbb{C})$.

A natural question is the following:

1.5. Question. — *Is this isomorphism induced by a natural isomorphism*

$$H^n_{\mathrm{dR}}(Z_\mathbb{Q}) \cong H^n_B(X(\mathbb{C}),\mathbb{Q})$$

over $\mathbb{Q}$?

The answer is "no". Let me illustrate this by an example.

1.6. Example. — Let $Z = \mathbb{G}_m = \mathrm{Spec}(\mathbb{Q}[z,\frac{1}{z}])$ be the multiplicative group. (Note that this is a smooth but not projective. So is not an example of the type of varieties we are interested in, but it still illustrates, what is going on.) Then $Z_\mathbb{C} = \mathbb{C}^\times$. The first de Rham cohomology group is generated by $\omega = \frac{dz}{z}$:

$$H^1_{\mathrm{dR}}(Z_\mathbb{C}) \cong \mathbb{C}\cdot\omega.$$

The first singular Homology group is generated by the unit circle $\gamma = S^1$:

$$H_1(Z(\mathbb{C}),\mathbb{C}) \cong \mathbb{C}\cdot\gamma.$$

It is now easy to compute for the pair (ω, γ) the pairing from de Rham's theorem:

$$(\omega, \gamma) \mapsto \int_\gamma \omega = \int_1^{e^{2\pi i}} \frac{dz}{z} = \int_0^{2\pi} \frac{de^{i\theta}}{e^{i\theta}} = 2\pi i$$
$$= p^n \int_1^{e^{\frac{2\pi i}{p^n}}} \frac{dz}{z} = p^n \int_0^{\frac{2\pi}{p^n}} \frac{de^{i\theta}}{e^{i\theta}} = 2\pi i.$$

The comparison isomorphism $H_B^1(Z(\mathbb{C}), \mathbb{C}) \cong H^1_{\mathrm{dR}}(Z_\mathbb{C})$ is given by

$$\gamma^* \mapsto \Big(\int_\gamma \omega\Big)^{-1} \frac{dz}{z} = \frac{1}{2\pi i} \frac{dz}{z}.$$

1.7. Remark. — In general $H^n_{dR}(Z_\mathbb{Q})$ and $H^n_B(Z(\mathbb{C}), \mathbb{Q})$ are two different $\mathbb{Q}$-lattices in $H^n_{\mathrm{dR}}(Z_\mathbb{Q}) \cong H^n_B(X(\mathbb{C}), \mathbb{Q})$.

1.8. Remark. — Elements in the image of the non-degenerate pairing of de Rham's theorem are called **periods**. de Rham's theorem tells us that $\mathbb{C}$ contains all periods of algebraic varieties over $\mathbb{Q}$ and as such is "big enough".

2. The Hodge decomposition

The next step in Hodge theory, which was discussed in last times seminar is the Hodge theorem.

2.1. Theorem. — *Let $Z/\mathbb{Q}$ be a smooth projective algebraic variety. There is a natural decomposition*

$$H^n_{\mathrm{dR}}(Z_\mathbb{C}) \cong \bigoplus_{i+j=n} H^{i,j}(Z_\mathbb{C}) = \bigoplus_{i+j=n} H^i(Z_\mathbb{C}, \Omega^j_{Z_\mathbb{C}/\mathbb{C}}),$$

with $\overline{H^{i,j}(Z_\mathbb{C})} = H^{j,i}(Z_\mathbb{C})$ for the complex conjugation.

Together with de Rham's theorem, we obtain a Hodge decomposition of singular cohomology:

2.2. Corollary. — *There is a decomposition*

$$H^n_B(Z(\mathbb{C}), \mathbb{C}) \cong \bigoplus_{i+j=n} H^i(Z_\mathbb{C}, \Omega^j_{Z_\mathbb{C}/\mathbb{C}}).$$

2.3. Remark. — One can interpret this as follows: it is possible to recover the action of the absolute Galois group $G_\mathbb{R} = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$, which is just the complex conjugation, on the Betti cohomology from de Rham cohomology.

2.4. Remark. — This has some powerful applications. For example, if $Z = E$ is an elliptic curve, singular cohomology and de Rham-cohomology together with the Hodge decomposition encode the isomorphy type of E.

2.5. Question. — *To what extend can such theorems be true for other (types of) fields?*

3. Completions of the rational numbers

Up to now we started with a variety over $\mathbb{Q}$ and as we found out, we had to work with complex coefficients.

3.1. Construction. — We all know that we can obtain the complex numbers $\mathbb{C}$ from the rational numbers $\mathbb{Q}$ by the following procedure:

(i) Complete $\mathbb{Q}$ with respect to the usual absolute value, i.e. the natural archimedean value $|\cdot|$.

(ii) Take the algebraic closure. This remains complete with respect to $|\cdot|$.

The process is illustrated by the picture

$$\mathbb{Q} \hookrightarrow \widehat{\mathbb{Q}} = \mathbb{R} \hookrightarrow \overline{\mathbb{R}} = \mathbb{C}.$$

To understand the picture completely, we want to consider also non-archimedean completions of $\mathbb{Q}$.

3.2. ***Definition.*** — Let p be a prime number. Every rational number $X \neq 0$ can be written in the form $x = \pm\frac{a}{b}p^n$, with $n \in \mathbb{Z}$ and $a, b \in \mathbb{N}$ relative prime to p. The **p-adic norm** is defined by

$$|x|_p := p^{-n} \qquad \text{and} \qquad |0|_p = 0.$$

It satisfies

— multiplicativity: $|xy|_p = |x|_p \cdot |y|_p$,

— the strong triangle inequality: $|x+y|_p \leqslant \max(|x|_p, |y|_p)$.

3.3. ***Remark.*** — According to Ostrowski's theorem every non-trivial norm on $\mathbb{Q}$ is equivalent to the natural archimedean norm or a p-adic one for some prime p.

3.4. ***Example.*** — For $|\cdot|_p$ high powers of p become small.

— Let $p = 3$. Then

$$\left|\frac{28}{3}\right|_3 = \left(\frac{1}{3}\right)^{-1} = 3,$$
$$|2000|_3 = \left(\frac{1}{3}\right)^{0} = 1,$$
$$\left|\frac{3^6}{2}\right|_3 = \left(\frac{1}{3}\right)^{6} = \frac{1}{2187}$$

— A series $\sum_{n=0}^{\infty} a_n$ with $a_n \in \mathbb{Q}$ converges p-adically, if $|a_n|_p \to 0$. For example the geometric series $\sum_{n=0}^{\infty} p^n$ converges p-adically

$$\sum_{n=0}^{\infty} p^n = \frac{1}{1-p}$$

Let us now fix some prime p.

3.5. ***Construction.*** — As above, we complete the rational numbers:

(i) Complete $\mathbb{Q}$ with respect to the p-adic norm $|\cdot|_p$.

(ii) Pass to the algebraic closure. Note that in contrast to the complex numbers this is not complete with respect to $|\cdot|_p$.

(iii) Thus we have to complete again. Now we have obtained something that is algebraically closed and complete.

As above, the process is illustrated by the following diagram

$$\mathbb{Q} \hookrightarrow \widehat{\mathbb{Q}} = \mathbb{Q}_p \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow \widehat{\overline{\mathbb{Q}}}_p = \mathbb{C}_p.$$

3.6. ***Remark.*** — (i) The p-adic integers are given by

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid |x|_p \leqslant 1\}.$$

We have $\mathbb{Z}_p \cong \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$ and $\mathbb{Q}_p = \mathbb{Z}_p\left[\frac{1}{p}\right]$.

(ii) Another difference to the classical situation is, that the absolute Galois group $G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ is huge in comparison to $G_{\mathbb{R}} := \mathrm{Gal}(\mathbb{C}/\mathbb{R})$: $\overline{\mathbb{Q}}_p$ is infinite dimensional over $\mathbb{Q}_p$, as $x^n - p$ is irreducible in $\mathbb{Q}_p[x]$.

(iii) We have $G_{\mathbb{Q}_p} = \mathrm{Aut}_{\mathrm{cont}}(\mathbb{C}_p)$. The dimension $\dim_{\mathbb{Q}_p}(\mathbb{C}_p)$ is not even countable. The axiom of choice produces an isomorphism of abstract fields $\mathbb{C}_p \cong \mathbb{C}$.

Now we want to consider the following question:

3.7. Question. — *Is there a p-adic analogue of the above comparison theorems?*

4. Building blocks for p-adic comparison theorems

Let again $Z/\mathbb{Q}$ be a smooth projective algebraic variety. To obtain analogues of the de Rham and the Hodge theorem, we need analogues of the cohomology theories involved.
Of course we can also take the algebraic de Rham cohomology with coefficients in $\mathbb{Q}_p$:

$$H^*_{\mathrm{dR}}(Z_{\mathbb{Q}_p}).$$

But we also need a cohomology that behaves like Betti cohomology but with coefficients in $\mathbb{Q}_p$.

4.1. Definition. — Let

$$H^*_{\text{ét}}(Z_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) := \left(\varprojlim H^*_{\text{ét}}(Z_{\overline{\mathbb{Q}}_p}, \mathbb{Z}/p^n\mathbb{Z})\right) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

be Grothensieck's étale cohomology.

4.2. Remark. — This can be seen as an analogue of Betti cohomology as there is an isomorphism

$$H^*_B(X(\mathbb{C}), \mathbb{Q}) \times_{\mathbb{Q}} \mathbb{Q}_p \cong H^*_{\text{ét}}(Z_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$$

(by choosing an embedding $\mathbb{Q}_p \hookrightarrow \mathbb{C}$).

4.3. Remark. — It has the following properties:

(i) The vector spaces $H^n_{\text{ét}}(Z_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ are finite dimensional over $\mathbb{Q}_p$. In fact, by the above isomorphism, they have the "correct" dimension.

(ii) The come with a continuous action of $G_{\mathbb{Q}_p}$ (coming from the natural action on $X_{\overline{\mathbb{Q}}_p}$).

5. Heuristic: a p-adic integral

The first natural question to ask is the following:

5.1. Question. — *Is there a canonical isomorphism of $H^*_{\text{ét}}(Z_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p$ and $H^*_{\mathrm{dR}}(Z_{\mathbb{C}_p})$?*

For this let us consider the following example

5.2. Example. — Let again be $Z = \mathbb{G}_m = \mathrm{Spec}(\mathbb{Q}[z, \frac{1}{z}])$.

— The first de Rham cohomology group is generated by $\omega = \frac{dz}{z}$: $H^1_{dR}(Z_{\mathbb{Q}_p}) \cong \mathbb{Q}_p \cdot \frac{dz}{z}$.

— A natural replacement for the first singular homology group is the Tate module: We have the cyclotomic character $\chi : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^*$, defined via

$$\sigma(e^{\frac{2\pi i}{p^n}}) = e^{\chi(\sigma)\frac{2\pi i}{p^n}}, n \geqslant 1.$$

Or more precisely: Let ζ_n be a primitive $p^{n\text{th}}$ root of unity. Any $p^{n\text{th}}$ root of unity corresponds to an element of $\mathbb{Z}/p^n\mathbb{Z}$. The primitive one's correspond to elements in $(\mathbb{Z}/p^n\mathbb{Z})^\times$. Every $p^{n\text{th}}$ root of unity is a power of ζ_n. An element $\sigma \in G_{\mathbb{Q}_p}$ sends ζ_n to another primitive $p^{n\text{th}}$ root of unity: $\sigma(\zeta_n) = \zeta_n^{a_{\sigma,n}}$ with $a_{\sigma,n} \in (\mathbb{Z}/p^n\mathbb{Z})^\times$. The cyclotomic character is defined by

$$\chi : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times, \sigma \mapsto (a_{\sigma,n})_n.$$

For $r \in \mathbb{Z}$, denote by $\mathbb{Q}_p(r)$ the r^{th} Tate twist: it is $\mathbb{Q}_p$ with the action of $G_{\mathbb{Q}_p}$ given by χ^r. We have

$$\mathbb{Q}_p(1) \cong \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \varinjlim \mathbb{G}_m(\overline{\mathbb{Q}}_p)[p^n] = V_p\mathbb{G}_m.$$

Cohomologically

$$V_p\mathbb{G}_m \cong H^1_{\text{ét}}(\mathbb{G}_{m,\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)^*.$$

It is generated by an element $\widehat{\epsilon} := (\epsilon_n)_n$, with $\epsilon_n \in \overline{\mathbb{Q}}_p$ satisfying $\epsilon_n^p = \epsilon_{n-1}$ for $n > 1$, and $\epsilon_1^p = 1$ and $\epsilon_1 \neq 1$.

There are theories of the p-adic integral (Coleman, Colmez) for which the following calculation makes sense:

$$\int_\epsilon \omega = \int_1^{\widehat{\epsilon}} \frac{dz}{z} = p^n \int_1^{\epsilon_n} \frac{dz}{z} = p^n \log_p(\epsilon_n) = \log_p(\epsilon_n^{p^n}) = \log_p(1) = 0.$$

And of course this is not, what we would like to have to obtain a non-degenerate pairing.
A possible interpretation is, that there is no analogue of $2\pi i$ in $\mathbb{C}_p$. This was made precise by Tate (1966). In other words, $\mathbb{C}_p$ does not have enough periods.

5.3. Remark. — The p-adic logarithm is defined as follows:
The goal is to construct a continuous $\log_p : \mathbb{C}_p^\times \to \mathbb{C}_p$, such that $\log_p(xy) = \log_p(x) + \log_p(y)$. Since $\mathbb{C}_p^* = p^{\mathbb{Q}} \times \mu \times U^{(1)}$, where μ is the roots of unity of order prime to p, and $U^{(1)}$ is the group of principal units. Then it suffices to define $\log_p$ on each of the factors. On $U^{(1)}$ it is defined by the usual power series $\log_p(x) = -\sum_{n\geqslant 1} \frac{(1-x)^n}{n}$. On μ it has to be zero, because for any root of unity u of order n, $n \cdot \log_p(u) = \log_p(1) = 0$. It remains to determine the value for p. Since $\sigma \in G_{\mathbb{Q}_p}$ extends to a continuous automorphism of $\mathbb{C}_p$, it follows that $\log_p(p) \in \mathbb{Q}_p$. The simplest choice is $\log_p(p) = 0$. There are other branches of the p-adic logarithm, that depend upon a finite extension of $\mathbb{Q}_p$ and a choice of element in the maximal ideal of the ring of integers.

The question now is:

5.4. Question. — *Is there a p-adic ring $\mathbb{B}$, which contains periods for all Z varieties over a finite extensin $K/\mathbb{Q}_p$, such that*

(i) there is an isomorphism

$$H^n_{\text{dR}}(Z(\otimes_K \mathbb{B} \cong H^n_{\text{ét}}(Z_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B},$$

(ii) one can recover the Galois representation on the étale cohomology $H^n_{\text{ét}}(Z_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ from the de Rham cohomology?

As we will see, the answer is 'yes", but with (relatively) complicated period rings introduced by Fontaine.

6. The de Rham period ring

The first approximation for such a ring was the de Rham period ring $\mathbb{B}^+_{\text{dR}}$ constructed by Fontaine circa 1980. In some sense, Fontaine added artificially the analogue of $2\pi i$ to $\mathbb{C}_p$. More precisely, it contains a distinguished element t, such that $G_{\mathbb{Q}_p}$ acts on $\mathbb{B}^+_{\text{dR}}$ and it acts on t via the cyclotomic character: $\sigma(t) = \chi(\sigma)t$. It has the following properties:

(i) $\mathbb{B}^+_{\text{dR}} \cong \mathbb{C}_p[\![t]\!]$, but not in any reasonable way. (There is a K-linear continuous section $\mathbb{C}_p \to \mathbb{B}^+_{\text{dR}}$ of the surjective map $\theta : \mathbb{B}^+_{\text{dR}} \to \mathbb{C}_p$, but not preserving the ring structure. On the other hand, the axiom of choice gives sections preserving the ring structure but they cannot be continuous.)

(ii) There is however a short exact sequence

$$0 \to t\mathbb{B}^+_{\text{dR}} \to \mathbb{B}^+_{\text{dR}} \xrightarrow{\theta} \mathbb{C}_p \to 0.$$

(iii) $\mathbb{B}_{\mathrm{dR}}^+$ is equipped with a descending filtration by the powers of t:

$$\mathbb{B}_{\mathrm{dR}}^+ \supset F^n\mathbb{B}_{\mathrm{dR}}^+ := (t^n)$$

with graded pieces

$$\mathrm{gr}_F^n\mathbb{B}_{\mathrm{dR}}^+ \cong \mathbb{C}_p(n).$$

(iv) $\mathbb{B}_{\mathrm{dR}}^+$ is a completion of $\overline{\mathbb{Q}}_p$ involving "higher derivatives". This was made precise by Colmez.

6.1. Definition. — Define now $\mathbb{B}_{\mathrm{dR}} := \mathbb{B}_{\mathrm{dR}}^+\left[\frac{1}{t}\right]$.

***6.2. Theorem* (Faltings, 1989).** — *Let Z/K be be a proper smooth variety, $[K:\mathbb{Q}_p] < \infty$. There is an isomorphism*

$$\alpha_{\mathrm{dR}} : H_{\mathrm{dR}}^n(Z) \otimes_K \mathbb{B}_{\mathrm{dR}} \cong H_{\text{ét}}^n(Z_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{dR}}$$

compatible with the Galois action and the filtration, where the Hodge filtration on de Rham cohomology is defined by

$$F^i H_{\mathrm{dR}}^n(Z) := \mathrm{Im}\,(H^n(Z, \Omega_{Z/K}^{\geqslant i}) \to H_{\mathrm{dR}}^n(Z)).$$

6.3. Remark. — Similar to the classical case, one can extend this theorem to all K-varieties using alterations instead of resolutions of singularities. Hence one can say that $\mathbb{B}_{\mathrm{dR}}$ contains periods for all algebraic varieties over K.

6.4. Remark. — The de Rham comparison theorem yields a filtered isomorphism obtained by taking G_K-fixed points

$$H_{\mathrm{dR}}^n(Z) \cong (H_{\text{ét}}^n(Z_{\overline{K}}, \mathbb{Q}_p) \otimes \mathbb{B}_{\mathrm{dR}}^{G_K}.$$

Thus it is possible to recover $H_{\mathrm{dR}}^*(Z)$ from $H_{\text{ét}}^*(Z_{\overline{K}}, \mathbb{Q}_p)$.
But: we cannot go the other way. The reason is, that the structure both on de Rham cohomology and on $\mathbb{B}_{\mathrm{dR}}$ is too coarse. All we have, is the Hodge filtration ....

However, we can use it, to obtain as a corollary an analogue of the Hodge decomposition. It was conjectured by Tate and the starting point for p-adic Hodge theory:

6.5. Corollary. — *There is a Galois equivariant decomposition*

$$H_{\text{ét}}^n(Z_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{i+j=n} H^j(Z, \Omega_{Z/K}^i) \otimes_K \mathbb{C}_p(-i).$$

It says that the twist of the Galois representation by $\mathbb{C}_p$ splits as a direct sum of cyclotomic characters with multiplicities given by the Hodge numbers.

7. Refinements

We have seen, that we need additional data on the left hand side. To recover the Galois representation on étale cohomology, Fontaine defined more refined period rings:

$$\mathbb{B}_{\mathrm{cris}} \subset \mathbb{B}_{\mathrm{st}} \subset \overline{K} \cdot \mathbb{B}_{\mathrm{st}} \subset \mathbb{B}_{\mathrm{dR}}.$$

(i) The crystalline period ring $\mathbb{B}_{\mathrm{cris}}$ is equippes with (commuting) G_K-action and a Frobenius operator φ.

(ii) The semistable period ring has, besides the G_K-action and the Frobenius φ, a $\mathbb{B}_{\mathrm{cris}}$-linear monodromy operator N, which commutes with the G_K-action and satisfies $N\varphi = p\varphi N$, such that $\mathbb{B}_{\mathrm{st}}^{N=0} = \mathbb{B}_{\mathrm{cris}}$ (in other words, it is a $\mathbb{B}_{\mathrm{cris}}$-derivation).

7.1. Remark. — All period rings are cohomologies of geometric points:

(i) We have $\mathbb{B}_{\mathrm{dR}}^+/F^m = (R\Gamma_{\mathrm{dR}}(\mathrm{Spec}(\mathscr{O}_{\overline{K}})) \otimes \mathbb{Q}_p)/F^m$, the derived log-de Rham cohomology.

(ii) We have $\mathbb{B}_{\mathrm{cris}}^{+} = R\Gamma_{\mathrm{cris}}(\mathrm{Spec}(\mathscr{O}_{\overline{K},1})) \otimes \mathbb{Q}_p$, the absolute crystalline cohomology. The Frobenius comes from the geometric Frobenius on $\mathscr{O}_{\overline{K},1} = \mathscr{O}_{\overline{K}}/p$.

(iii) We have $\mathbb{B}_{\mathrm{st}}^{+} = R\Gamma_{\mathrm{cris}}(\mathrm{Spec}(\mathscr{O}_{\overline{K},1}^{\times})/R^{\times}) \otimes \mathbb{Q}_p$, the log-crystalline cohomology (where the log structure on $\mathrm{Spec}(\mathscr{O}_{\overline{K},1}^{\times})$ is induced by the uniformiser, and $R^{\times}$ corresponds to the log-crystalline affine line, more precisely, it is the PD-envelope of $w(k)[x]$ with log structure induced by x). Again, the Frobenius is induced by the geometric Frobenius, and the monodromy is the Gauß–Manin connection.

To obtain additional structure on the de Rham cohomology, Fontaine and Jannsen conjectured the existence of a certain p-adic cohomology, with a similar structure. The first version of it was constructed by Hyodo and Kato, so it is now known as Hyodo–Kato theory, $H_{\mathrm{HK}}^{*}(Z)$. I will say more about it in the next part. It comes with

— a Frobenius φ,
— a monodromy operator N, such that $N\varphi = p\varphi N$,
— a G_K action and
— a canonical isomorphism $H_{\mathrm{HK}}^{*}(Z) \otimes_{K^{nr}} \overline{K} \cong H_{\mathrm{dR}}^{*}(Z_{\overline{K}})$.

8. Comparison theorem

Now it was possible to formulate the p-adic comparison theorems:

C_{cris} the crystalline conjecture due to Fontaine
C_{st} the semistable conjecture due to Fontaine–Jannsen.

8.1. Remark. — They are now theorems due to a number of people:
Fontaine–Messing, Hyodo, Kato, Faltings, Tsuji, Nizioł (1985 – 2005);
Beilinson, Bhatt, Scholze (2010+)

The most general version can be formulated as follows:

8.2. Theorem. — *Let Z be a K-variety. There is an isomorphism*

$$\alpha_{\mathrm{st}} : H_{\mathrm{HK}}^{n}(Z_{\overline{K}}) \otimes_{K^{nr}} \mathbb{B}_{\mathrm{st}} \cong H_{\mathrm{\acute{e}t}}^{n}(Z_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{st}}$$

compatible with Frobenius, monodromy, the Galois action and the de Rham period isomorphism α_{dR}

8.3. Remark. — Using the Hyodo–Kato map $H_{\mathrm{HK}}^{*}(Z) \otimes_{K^{nr}} \overline{K} \cong H_{\mathrm{dR}}^{*}(Z_{\overline{K}})$, we can say that $\overline{K} \cdot \mathbb{B}_{\mathrm{st}}$ contains periods for all algebraic K-varieties.

8.4. Remark. — Now we can also recover the étale cohomology including the Galois action:

$$H_{\mathrm{\acute{e}t}}^{n}(Z_{\overline{K}}, \mathbb{Q}_p) \cong (H_{\mathrm{HK}}^{n}(Z_{\overline{K}}) \otimes_{K^{nr}} \mathbb{B}_{\mathrm{st}})^{N=0,\varphi=1} \cap F^0(H_{\mathrm{dR}}^{n}(Z_{\overline{K}}) \otimes_{\overline{K}} \mathbb{B}_{\mathrm{dR}}).$$

Next time, I want to tell you more about the construction of Hyodo–Kato theory.

PART II. RIGID ANALYTIC HYODO–KATO THEORY

In this part, we want to focus on the additional structure that is needed on de Rham cohomology to make the comparison theorems work.

Notation. — I will use the following notation:

$$\begin{aligned} V & \quad - \quad \text{complete discrete valuation ring of mixed characteristic } (0,p); \\ \mathfrak{m} & \quad - \quad \text{its maximal ideal;} \\ \pi & \quad - \quad \text{a uniformiser of } V; \\ K & \quad - \quad \text{its fraction field;} \\ k & \quad - \quad \text{its residue field, which is perfect;} \\ W(k) & \quad - \quad \text{the ring of Witt vectors of } k; \\ K_0 & \quad - \quad \text{the fraction field of } W(k). \end{aligned}$$

For a scheme X/V we denote by

$$\begin{aligned} X_n & \quad - \quad \text{for } n \in \mathbb{N}, \text{ the reduction of } X \text{ modulo } p^n; \\ X_0 & \quad - \quad \text{its special fibre;} \\ X_K & \quad - \quad \text{its generic fibre.} \end{aligned}$$

Remark. — I want to focus on the case that X has semistable reduction. On the one hand, the smooth reduction case can evidently seen as a special case of it. On the other hand, it turns out that the general case can be reduced to the semistable case.
Recall that in the complex case, we said, that we can obtain the general case by using resolutions of singularities. In the mixed characteristic and positive characteristic case, this is in general not possible. However, one can work with alterations instead.

9. Hyodo–Kato theory

Let us first recall what we mean if we say "Hyodo–Kato theory".

9.1. Definition. — By a Hyodo–Kato theory for V-schemes X (or K-varieties), we mean

(i) a cohomology theory $H^*_{\mathrm{HK}}(X)$ in finite dimensional K_0-vector spaces;

(ii) a bijective Frobenius-linear operator $\varphi : H^*_{\mathrm{HK}}(X) \to H^*_{\mathrm{HK}}(X)$, called **Frobenius**.

(iii) a nilpotent operator $N : H^*_{\mathrm{HK}}(X) \to H^*_{\mathrm{HK}}(X)$ such that $N\varphi = p\varphi N$, called the **monodromy**.

(iv) a functorial morphism $\Psi : H^*_{\mathrm{HK}}(X) \to H^*_{\mathrm{dR}}(X_K)$, which is an isomorphism after tensoring with K, called the **Hyodo–Kato morphism**.

It is highly non-trivial to obtain Hyodo–Kato morphism. There are several constructions:

— Hyodo–Kato's original construction based on log crystalline cohomology. The Hyodo–Kato morphism Ψ_π^{HK} depends on the choice of a uniformiser π of V.

— Beilinson's representation of the Hyodo–Kato complex with a Hyodo–Kato morphism Ψ^B **independent** of the choice of a uniformiser.

— Große-Klönne's rigid analytic construction, using dagger spaces. The Hyodo–Kato map Ψ_π^{GK} depends on the choice of a uniformiser and is a zigzag through rather complicated intermediate objects.

Our goal was to obtain a Hyodo–Kato theory that **lends itself for computations** and is **indepenent of the choice of a uniformiser**.

9.2. Construction. — *(Ertl–Yamada) Let X/V be of semistable reduction. Using weak formal schemes and dagger spaces, we obtain*

— *a new presentation of the Hyodo–Kato cohomology*

$$R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(X)$$

together with a Frobenius φ and monodromy operator N;

— *a natural functorial morphism*

$$\Psi : R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(X) \to R\Gamma_{\mathrm{dR}}(X_K)$$

which is a quasi-isomorphism after tenosring with K. It has the following advantages:

— *It is* ***not*** *a zigzag.*
— *It is independent of the choice of a uniformiser.*
— *It is suitable for computations*

10. The classical construction

I will first explain the classical construction due to Hyodo and Kato. In this context log schemes play an important role.

10.1. Remark. — A log structure is a datum added to a scheme. Log schemes are often used when dealing with singularities, that are nice enough. They allow to generalise definitions and techniques from classical schemes. For example, there is a notion of log smoothness that extends the definition of classical smoothness. A classical example, where log structures are used is the following:

Example. — Let $X/\mathbb{C}$ be a smooth variety of dimension n, and $D \subset X$ a reduced normal crossings divisor. On a suitable polydisc B, we have

$$\begin{array}{ccccc} D \cap B & \hookrightarrow & B & & \\ \| \wr & & \| \wr & & \\ \{z_1 \cdots z_r = 0\} & \hookrightarrow & \{|z_i| < 1\} & \hookrightarrow & \mathbb{C} \end{array}$$

Define the sheaf of differential forms on X with logarithmic poles along D, denoted by $\Omega^1_X(\log D)$, as the sheaf of meromorphic 1-forms on X that are holomorphic away from D and on polydisks B as above can be written as

$$\sum_{i \leqslant r} f_i \frac{dz_i}{z_i} + \sum_{i > r} f_i dz_i, \text{ with all } f_i \text{ holomorphic.}$$

In order to generalise this, we use the following:

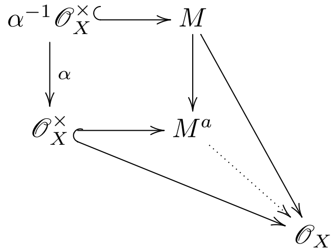
Definition. — A *pre-log structure* M on a scheme X is a pair (M, α) where M is a sheaf of monoids on the étale site $X_{\text{ét}}$ and $\alpha : M \to \mathscr{O}_{X_{\text{ét}}}$ is a homomorphism of sheaves of monoids. A *log structure*, is a pre-log structure, such that α induces an isomorphism

$$\alpha^{-1}\mathscr{O}_X^\times \xrightarrow{\sim} \mathscr{O}_X^\times$$

Some examples are:

— The initial object among all log structures is the trivial log structure $(\mathscr{O}_X^\times, \iota : \mathscr{O}_X^\times \to \mathscr{O}_X)$.
— The final object (which is of little use) is $(\mathscr{O}_X, \mathrm{id})$.

— If M is a pre-log structure on X, we can consider the associated log structure M^a, which is the push-out



in the category of sheaves of monoids.

— The log structure associated to a divisor as in the above example is given as follows: denote the closed immersion $i : D \hookrightarrow X$, and the open immersion $j : U := X \backslash D \hookrightarrow X$. Then the set $M_D = j_* \mathscr{O}_U^\times \cap \mathscr{O}_X$. In local coordinates, there is a so-called chart given by

$$\mathbb{N}_X^r \to \mathscr{O}_X ; (n_i) \mapsto \prod z_i^{n_i}.$$

This defines a pre-log structure, and we have to consider the associated log-structure as explained above.

Now we can define the complex of log differentials:

Definition. — More generally, for a morphism of log schemes $f : (X, M) \to (Y, N)$, one defines the sheaf of differential forms with logarithmic poles relative to f, denoted by $\mathscr{O}_X$-module $\omega^1_{(X,M)/(Y,N)}$, as the quotient of $\Omega^1_{X/Y} \oplus (\mathscr{O}_X \otimes_{\mathbb{Z}} M^{gp})$ by the submodule generated by $(d\alpha(m), 0) - (0, \alpha(m) \otimes m)$, for $m \in M$, and $(0, 1 \otimes n)$, for $a \in f^{-1}(N)$. The class of $(0, 1 \otimes m)$, for $m \in M$ is denoted by $d\log(m)$.

Some important notions in the context of log schemes:

Definition. — Let (X, M) be a log scheme.

— A pre- log structure is *integral*, if it is a sheaf of integral monoids. Then the associated log structure is integral.

— A log scheme (X, M) is *fine*, if it is integral and coherent. *Coherence* means that étale locally on X, there is a finitely generated monoid P and a homomorphism $\beta : P_X \to \mathscr{O}_X$, such that (M, α) is isomorphic to $(P_X, \beta)^a$. So the above standard example is a fine log scheme. We call this a chart of M. One can also define a chart of a morphism of log schemes.

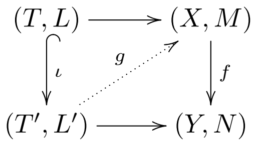
Now we can define the log-analogon of smoothness.

Definition. — Let $f : (X, M) \to Y, N)$ be a morphism of log schemes.

— A map of log schemes is called *strict*, if the associated map of monoids is an isomorphism.

— The morphism f is a *strict closed immersion*, if the map of underlying schemes is a closed immersion, and it is strict as a log map.

— If f is a morphism of fine log schemes, it is called *log-smooth* (resp, *log-étale*), if the underlying map of schemes is locally of finite presentation and for every commutative diagram of fine log schemes



where ι is a strict closed immersion and T is defined by a nilpotent ideal in $\mathscr{O}_{T'}$, there exists étale locally on T' a log map g (resp. there exists a unique log map g), such that the diagram commutes.

— Assume now in addition that X and Y are defined over $\mathbb{F}_p$. Then they have an absolute Frobenius map $F_{(X,M)}$ (the absolute Frobenius on the underlying scheme and the p^{th}-power map on monoids). We say f is of *Cartier type*, if it is integral, and in the following commutative diagram with a Cartesian square

$$\begin{array}{ccccc} & & \overset{F_{(X,M)}}{\frown} & & \\ (X,M) & \xrightarrow[g]{} & (X',M') & \xrightarrow[h]{} & (X,M) \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ & & (Y,N) & \xrightarrow{F_{(Y,N)}} & (Y,N) \end{array}$$

the morphism g is exact.
(We say that a morphism f is exact, if the diagram

$$\begin{array}{ccc} f^{-1}(N) & \longrightarrow & M \\ \downarrow & & \downarrow \\ f^{-1}(N)^{gp} & \longrightarrow & M^{gp} \end{array}$$

is Cartesian.)
If f is smooth and of Cartier type, it induces a Cartier isomorphism

$$C^{-1} : \omega^q_{X'/Y} \xrightarrow{\sim} \mathscr{H}^q(\omega^\bullet_{X/Y}).$$

Hyodo and Kato extended the theory of crystalline cohomology to schemes with fine log structures. With the right definitions one basically obtains a consistent theory by just decorating everything with log. They use it to obtain the additional structure on the de Rham cohomology.

10.2. Construction. — *Let X/V be proper and of semistable reduction. Consider the following base log schemes:*

$$\begin{array}{rcl} k^0 & - & (\mathrm{Spec}(k), 1 \mapsto 0) \\ W(k)^0 & - & (\mathrm{Spec}(W(k)), 1 \mapsto 0) \\ W(k)^\varnothing & - & (\mathrm{Spec}(W(k)), \mathrm{triv}) \\ V^\sharp & - & (\mathrm{Spec}(V), \mathrm{can}) \\ \mathcal{S} & - & (\mathrm{Spwf}(W(k))[\![s]\!], 1 \mapsto s) \\ \mathcal{S}_{\mathrm{PD}} & - & \text{the PD-envelope of } (V_1^\sharp \hookrightarrow \mathcal{S}) \end{array}$$

They fit into a commutative diagram

(1)
$$\begin{array}{ccccc} & & k^0 & & \\ & {\scriptstyle i_0}\swarrow & \downarrow{\scriptstyle \tau} & \searrow{\scriptstyle i_\pi} & \\ W^0 & \xrightarrow[j_0]{} & \mathcal{S}_{\mathrm{PD}} & \xleftarrow[j_\pi]{} & V^\sharp. \end{array}$$

where j_0 and j_π are given by $s \mapsto 0$ and $s \mapsto \pi$ respectively, i_0 is the canonical embedding, $\tau := j_0 \circ i_0$ and i_π the unique morphism, such that $\tau = j_\pi \circ i_\pi$.
For X, we consider the canonical log structure (associated to the special fibre). This is fine, log smooth and of Cartier type over $V^\sharp$.

Then consider the log crystalline complexes:

$$R\Gamma_{\mathrm{cris}}(X/V^{\sharp}) := \mathrm{holim} R\Gamma_{\mathrm{cris}}(X_1/V_n^{\sharp}),$$
$$R\Gamma_{\mathrm{cris}}(X/\mathcal{S}_{PD},\pi) := \mathrm{holim} R\Gamma_{\mathrm{cris}}(X_1/\mathcal{S}_{PD,n},\pi),$$
$$R\Gamma_{\mathrm{HK}}^{\mathrm{cris}}(X,\pi) := R\Gamma_{\mathrm{cris}}(Y/W(k)^0) := \mathrm{holim} R\Gamma_{\mathrm{cris}}(Y/W_n(k)^0).$$

Of these, the first and the last one are the ones of interest to us, the second one is used to link the two.

— *Note that there is a canonical quasi-isomorphism*

$$\gamma\colon R\Gamma_{\mathrm{dR}}(X_K) \xrightarrow{\sim} R\Gamma_{\mathrm{cris}}(X/V^{\sharp})_{\mathbb{Q}},$$

where the left hand side is the de Rham cohomology of X_K with the Hodge filtration. So this gives us the link to de Rham cohomology.

— *The cohomology groups of $R\Gamma_{\mathrm{HK}}^{\mathrm{cris}}(X,\pi)$ are finite K_0-vector spaces.*

— *The Frobenius action φ on $R\Gamma_{\mathrm{HK}}^{\mathrm{cris}}(X,\pi)$ (respectively $R\Gamma_{\mathrm{cris}}(X/\mathcal{S}_{PD},\pi)$) is induced by the absolute Frobenius on Y (respectively X_1) and the Frobenius σ on W (respectively on $\mathcal{S}_{PD}$). Hyodo–Kato showed that is invertible on $R\Gamma_{\mathrm{HK}}^{\mathrm{cris}}(X,\pi)_{\mathbb{Q}}$.*

— *The monodromy operator is defined as the boundary map of a certain short exact sequence. (It can also be described as a Gauß–Manin connection.)*

— *To obtain the Hyodo–Kato map, consider the morphisms*

$$R\Gamma_{\mathrm{HK}}^{\mathrm{cris}}(X,\pi)_{\mathbb{Q}} \xleftarrow{j_0^*} R\Gamma_{\mathrm{cris}}(X/\mathcal{S}_{PD},\pi)_{\mathbb{Q}} \xrightarrow{j_\pi^*} R\Gamma_{\mathrm{cris}}(X/V^{\sharp})_{\mathbb{Q}} \tag{2}$$

induced by the morphisms of log schemes j_0 and j_π. Hyodo–Kato showed that j_0^* admits (in the derived category) a unique functorial K_0-linear section $s_\pi\colon R\Gamma_{\mathrm{HK}}^{\mathrm{cris}}(X)_{\mathbb{Q}} \to R\Gamma_{\mathrm{cris}}(X/\mathcal{S}_{PD},\pi)_{\mathbb{Q}}$ which commutes with the Frobenius. (This section is itself a zig-zag and comes roughly from the observation that integrally Frobenius contracts. So if one twists integrally the scalars with a high enough power of the Frobenius (depending on the ramification) then one obtains a quasi-isomorphism rationally.) We set

$$\Psi_\pi^{\mathrm{cris}} := j_\pi^* \circ s_\pi\colon R\Gamma_{\mathrm{HK}}^{\mathrm{cris}}(X,\pi)_{\mathbb{Q}} \to R\Gamma_{\mathrm{cris}}(X/V^{\sharp})_{\mathbb{Q}}.$$

It induces a K-linear functorial quasi-isomorphism

$$\Psi_{\pi,K}^{\mathrm{cris}} := \Psi_\pi^{\mathrm{cris}} \otimes 1\colon R\Gamma_{\mathrm{HK}}^{\mathrm{cris}}(X,\pi) \otimes_{W(k)} K \to R\Gamma_{\mathrm{cris}}(X/V^{\sharp})_{\mathbb{Q}}.$$

11. Local description of our construction

Next I will explain our construction locally.

11.1. Notation. — We use similar base log schemes as above:

$$\begin{array}{rcl} k^0 & - & (\mathrm{Spec}(k), 1\mapsto 0) \\ W(k)^0 & - & (\mathrm{Spec}(W(k)), 1\mapsto 0) \\ W(k)^{\varnothing} & - & (\mathrm{Spec}(W(k)), \mathrm{triv}) \\ V^{\sharp} & - & (\mathrm{Spec}(V), \mathrm{can}) \\ \mathcal{S} & - & (\mathrm{Spwf}(W(k))[\![s]\!], 1\mapsto s) \end{array}$$

11.2. Remark. — Maybe you wonder about the "weak formal scheme $\mathcal{S}$. Strictly speaking, it is not a weak formal scheme, but rather a "pseudo-weak formal scheme" or a These are not necessarily adic over the base, but rather adic over a polynomial algebra over the base. In this sense, $W(k))[\![s]\!]$ is pseud-weakly complete finitely generated over $W(k)$. The scheme $\mathcal{S}$ is in fact the open unit disc and a non-p-adic weak formal scheme over $W(k)$.

11.3. Situation. — Now consider the situation

$$\begin{array}{rcl} Y & - & \text{semistable over } k^0; \\ \mathcal{Z} & - & \text{a lift to } \mathcal{S} \Rightarrow \text{log smooth over } W(k)^{\varnothing}; \\ \mathcal{X} & - & \mathcal{Z} \times V^{\sharp}; \\ \mathcal{Y} & - & \mathcal{Z} \times W(k)^0; \\ \mathfrak{Z}, \mathfrak{X}, \mathfrak{Y} & - & \text{the associated dagger spaces.} \end{array}$$

11.4. Construction. — Now we can compute different log rigid cohomologies

$$\begin{array}{rcl} \omega^{\bullet}_{\mathcal{Z}/W(k)^{\varnothing},\mathbb{Q}} & - & \text{computes the “absolute” rigid cohomolog } R\Gamma_{\mathrm{rig}}(Y/W(k)^{\varnothing}); \\ \omega^{\bullet}_{\mathcal{Z}^{\varnothing}/W(k)^{\varnothing},\mathbb{Q}} & - & \text{computes the non-logaritmic rigid cohomology } R\Gamma_{\mathrm{rig}}(Y^{\varnothing}/W(k)^{\varnothing}) = R\Gamma_{\mathrm{rig}}(Y/K_0); \\ \omega^{\bullet}_{\mathcal{X}/V^{\sharp},\mathbb{Q}} & - & \text{computes } R\Gamma_{\mathrm{rig}}(Y/V^{\sharp}); \\ \omega^{\bullet}_{\mathcal{Y}/W(k)^0,\mathbb{Q}} & - & \text{computes } R\Gamma_{\mathrm{rig}}(Y/W(k)^0)\text{; should give Hyodo–Kato theory;} \\ \widetilde{\omega}^{\bullet}_{\mathcal{Y},\mathbb{Q}} & - & \text{the auxiliary complex } \omega^{\bullet}_{\mathcal{Z}/W(k)^{\varnothing},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{Z}}} \mathcal{O}_{\mathfrak{Y}}; \end{array}$$

11.5. Proposition. — *We observed that $\widetilde{\omega}^{\bullet}_{\mathcal{Y},\mathbb{Q}}$ computes the same cohomology as $\omega^{\bullet}_{\mathcal{Z}/W(k)^{\varnothing},\mathbb{Q}}$, namely $R\Gamma_{\mathrm{rig}}(Y/W(k)^{\varnothing})$.*

11.6. Construction. — We now consider so called Kim–Hain complexes:

$$\omega^{\bullet}_{\mathcal{Z}/W(k)^{\varnothing},\mathbb{Q}}[u] \quad \text{and} \quad \widetilde{\omega}^{\bullet}_{\mathcal{Y},\mathbb{Q}}[u]$$

with $u^{[i]}$ of degree 0, such that $du^{[i+1]} = d\log s \cdot u^{[i]}$ and $u^{[0]} = 1$ and

— multiplication: $u^{[i]} \wedge u^{[j]} = \frac{(i+j)!}{i!j!} u^{[i+j]}$
— Frobenius: $\phi(u^{[i]}) = p^i u^{[i]}$
— monodromy: $N(u^{[i]}) = u^{[i-1]}$

11.7. Remark. — The idea of this construction goes back to Steenbrink when he studied limits of Hodge structures in the classical context. It was adapted by Mokrane to the crystalline setting, and then refined by Kim and Hain.
However, in the crystalline setting it only makes sense to consider the analogue of $\widetilde{\omega}^{\bullet}_{\mathcal{Y},\mathbb{Q}}[u]$. Moreover, the intention for using this construction had more to do with the monodromy, than with the Hyodo–Kato map.

11.8. Definition. — The rigid Hyodo–Kato cohomology for Y/k semistable is given by $R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y) := R\Gamma(\mathfrak{Z}, \omega^{\bullet}_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[u])$ with endomorphisms φ and N, such that $N\varphi = p\varphi N$.

This is justified by the following commutative diagram:

$$\begin{array}{ccccc} R\Gamma(\mathfrak{Z}, \omega^{\bullet}_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[u]) & \longrightarrow & R\Gamma(\mathfrak{Z}, \omega^{\bullet}_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[\![u]\!]) & \xrightarrow[u^{[i]}\mapsto 0]{\sim} & R\Gamma(\mathfrak{Z}, \omega^{\bullet}_{\mathcal{Z}/\mathcal{S},\mathbb{Q}}) \\ \downarrow \wr & & \downarrow & & \downarrow \\ R\Gamma(\mathfrak{Y}, \widetilde{\omega}^{\bullet}_{\mathcal{Y},\mathbb{Q}}[u]) & \xrightarrow{\sim} & R\Gamma(\mathfrak{Y}, \widetilde{\omega}^{\bullet}_{\mathcal{Y},\mathbb{Q}}[\![u]\!]) & \xrightarrow[u^{[i]}\mapsto 0]{\sim} & R\Gamma(\mathfrak{Y}, \omega^{\bullet}_{\mathcal{Y}/W^0,\mathbb{Q}}) \end{array}$$

11.9. Remark. — So we have indeed defined a cohomology of K_0-vector spaces, that has monodromy and Frobenius. From the diagram, it looks like we only used the complex $\widetilde{\omega}^{\bullet}_{\mathcal{Y},\mathbb{Q}}[u]$ to relate our definition to the cohomology of Y/W^0. So why did we use the complex $\omega^{\bullet}_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[u]$?
The reason is that it allows a straight forward definition of the Hyodo–Kato morphism. And this is a result of the first line of the diagram.

11.10. Definition. — We set

$$\begin{aligned} R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(\mathcal{X},\pi) &:= R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y); \\ R\Gamma_{\mathrm{dR}}(\mathcal{X}) &:= R\Gamma(\mathfrak{X}, \omega^{\bullet}_{\mathcal{X}/V^{\sharp},\mathbb{Q}}). \end{aligned}$$

and define for a uniformiser $\pi \in V$ and $q \in \mathfrak{m}\backslash\{0\}$

$$\Psi_{\pi,q} : R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(\mathcal{X},\pi) \to R\Gamma_{\mathrm{dR}}(\mathcal{X})$$

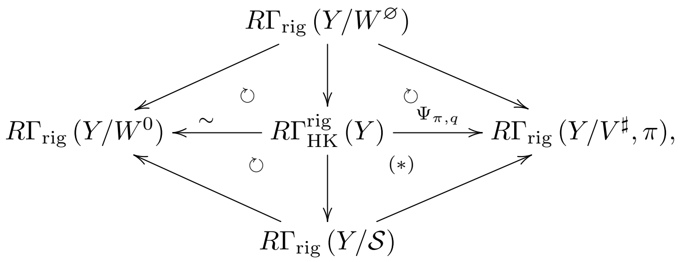
induced by the natural morphism $\omega^{\bullet}_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}} \to \omega^{\bullet}_{\mathcal{Z}/\mathcal{S},\mathbb{Q}} \to \omega^{\bullet}_{\mathcal{X}/V^{\sharp},\mathbb{Q}}$ and $\Psi_{\pi,q}(u^{[i]}) := \frac{(-\log_q(\pi))^i}{i!}$.

11.11. Remark. — Let $\log\colon V^{\times} \to K$ be the p-adic logarithm function defined by

$$\begin{aligned} \log(v) &:= -\sum_{n\geq 1} \frac{(1-v)^n}{n} \qquad \text{for } v \in (1+\mathfrak{m}), \\ \log(u) &:= 0 \qquad \text{for } u \in \boldsymbol{\mu}. \end{aligned}$$

A *branch of the p-adic logarithm* on K is a group homomorphism from $K^{\times}$ to (the additive group of) K whose restriction to $V^{\times}$ coincides with log as above.
For $q \in \mathfrak{m}\backslash\{0\}$, let $\log_q\colon K^{\times} \to K$ be the unique branch of the p-adic logarithm which satisfies $\log_q(q) = 0$. More precisely, for any uniformiser π the element q can be written as $q = \pi^m v$, for some $m \geq 1$ and $v \in V^{\times}$. Thus if we set $\log_q(\pi) := -m^{-1}\log(v)$, it extends to a group homomorphism $\log_q\colon K^{\times} \to K$.

So the diagram now looks like:



where all triangles except for $(*)$ commute. The triangle $(*)$ commutes if $q = \pi$.

11.12. Theorem. — *(Ertl–Yamada)*

(i) $\Psi_{\pi,q}$ *is independent o the choice of a uniformiser, that is for two uniformisers* $\pi, \pi' \in V$ *we have*

$$\Psi_{\pi,q} = \Psi_{\pi',q}.$$

(ii) It depends on the choice of a branch of the p-adic logarithm $\log_q$, *that is for* $q, q' \in \mathfrak{m}\backslash\{0\}$

$$\Psi_{\pi,q} = \Psi_{\pi,q'} \circ \exp\left(-\frac{\log_q(q')}{\mathrm{ord}_p(q')}N\right).$$

(iii) For any π *and* q,

$$\psi_{\pi,q} \otimes K : R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(\mathcal{X})_K \to R\Gamma_{\mathrm{dR}}(\mathcal{X}_K)$$

is a quasi-isomorphism.

(iv) For a choice of uniformiser $\pi \in V$, $\Psi_{\pi,\pi}$ *is compatible with the maps* Ψ^{HK}_{π} *of Hyodo–Kato and* Ψ^{GK}_{π} *of Große-Klönne.*

(v) If Y *has a compactification* $\overline{Y}$ *by a strictly semistable scheme with horizontal divisor, there is a rigid Hyodo–Kato theory of* Y *with compact support such that Poincaré duality is satisfied.*

11.13. Remark. — So far, we have these definitions also for a rather limited category of coefficients - so called unipotent coefficients (iterated extensions of the structure sheaf). But more general coefficients come up naturally (for example in the study of (p-adic) L-functions. So a natural question is, how to extend this to more general coefficients.
Another question is, how to define a theory with compact support in the case, when there is no "nice" compactification.

References

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