Integral p-adic cohomology theories for open and singular varieties

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  - concerns the Zeta function of a variety in positive characteristic p
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  - accomplished using ℓ-adic cohomology ($\ell \neq p$)
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- For all of this talk:
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  - $W(k)$ – ring of Witt vectors of k
  - $K = \mathrm{Frac}(W(k))$ – fraction field of $W(k)$

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- Candidates:
  - $H^*_{\mathrm{cris}}(X/W(k))$, Grothendieck/Berthelot
    Finitely generated over $W(k)$ **only for proper smooth X/k!**
  - $H^*_{\mathrm{rig}}(X/K)$, Monsky–Washnitzer (local), Berthelot (global)
    Is finitely generated over K, but has **rational coefficients!**
  - $H^*(X, W^\dagger\Omega^\bullet)$, Davis–Langer–Zink
    Compares rationally to rigid cohomology, but **not finitely generated over** $W(k)$ even modulo torsion! (Cunterexample: E–Shiho)

Question

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Good means ...

- The cohomology groups $H^*_{\text{good}}(X)$ are **finitely generate $W(k)$-modules** for all $X \in \mathrm{Var}_k$.
- There is a **comparison isomorphism with (log) crystalline cohomology** for $X \in \mathrm{Var}_k$ (log) smooth proper.
- There is **a rational comparison isomorphism with rigid cohomology** for all $X \in \mathrm{Var}_k$.

Some useful notions

- **Geometric pair** $(X, \overline{X})$**:** open immersion $X \hookrightarrow \overline{X}$ in Var_k with dense image, $\overline{X}$ proper – Var_k^{geo}
- **Normal crossing pair** $(X, \overline{X})$**:** geometric pair, such that $\overline{X} \backslash X$ is a simple normal crossing divisor – Var_k^{nc}
- **Morphisms of geometric pairs** $f : (X_1, \overline{X}_1) \to (X_2, \overline{X}_2)$**:** a morphism $f : \overline{X}_1 \to \overline{X}_2$ in Var_k such that $f(X_1) \subset X_2$
  - **strict:** if $f^{-1}(X_2) = X_1$
  - **has property P:** if $f : \overline{X}_1 \to \overline{X}_2$ has property P

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Conceptually ...

... normal crossing pairs are to geometric pairs, what smooth varieties are to varieties.

Hypothesis that we need

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Strong resolutions of singularities **(SR)**

- For all $X \in \mathrm{Var}_k$ there exists a proper birational morphism $f : X' \to X$, such that X' is smooth, and f is an iso on X_{sm}.
- For all proper birational morphisms $f : X' \to X$ in Sm_k there is a sequence of blow-ups along smooth centres

$$X_n \to X_{n-1} \to \cdots \to X$$

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Embedded resolutions of singularities **(EB)**

For all $(X, \overline{X}) \in \mathrm{Var}_k^{geo}$ with $\overline{X}$ smooth, there is a proper birational morphism $f : (X, \overline{X}') \to (X, \overline{X})$ such that $(X, \overline{X}') \in \mathrm{Var}_k^{nc}$.

Weak factorisation **(WF)**

For every strict proper birational morphism $(X, \overline{X}') \to (X, \overline{X})$ in Var_k^{nc} which is an iso on X, there exists a weak factorisation.

- A weak factorisation of a proper birational morphism $f : (X_1, \overline{X}_1) \to (X_2, \overline{X}_2)$ which is an iso on X_2 is a sequence

$$(X_1, \overline{X}_1) = (V_0, \overline{V}_0) \overset{f_1}{\dashrightarrow} \ldots \overset{f_\ell}{\dashrightarrow} (V_\ell, \overline{V}_\ell) = (X_2, \overline{X}_2)$$

where f_i is rational, $f_\ell \circ \ldots \circ f_1 = f$, each composition $f_i \circ \ldots \circ f_1$ is a morphism and induced an iso on X_2, and for each i either f_i or f_i^{-1} is a blow-up along a smooth center Z_i disjoint from X_2 which has normal crossing with $\overline{V}_i \backslash V_i$ (or $\overline{V}_{i-1} \backslash V_{i-1}$.

Embedded resolutions with boundaries **(ERB)**

For all strict closed immersions of geometric pairs $(Y, \overline{Y}) \to (X, \overline{X})$ such that Y is smooth and $(X, \overline{X}) \in \mathrm{Var}_k^{nc}$, there is a commutative diagram

$$\begin{array}{ccc} (Y, \overline{Y}') & \hookrightarrow & (X, \overline{X}') \\ \downarrow & & \downarrow \\ (Y, \overline{Y}) & \hookrightarrow & (X, \overline{X}) \end{array}$$

where the horizontal maps are strict closed immersions and the vertical maps are proper birational morphisms and isos on Y, X.

Topologies on Var_k and Sm_k

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The *cdp*-topology on Var_k ...

... is the topology generated by completely decomposed proper morphisms $p : Y \to X$.
(Completely decomposed means that for every $x \in X$ there is $y \in p^{-1}(x) \subset Y$, such that for the residue fields $\kappa(x) \xrightarrow{\sim} \kappa(y)$.)

Lemma (Suslin–Voevodsky)

The cdp-topoology on Var_k *is generated by blow-ups.*

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Lemma (Suslin–Voevodsky)

The cdp-topoology on Var_k *is generated by blow-ups.*

The *rh*-topology on Var_k ...

... is the topology generated by *cdp*-morphisms and Zariski morphisms.

One can restrict these topologies to Sm_k

Proposition (E–Shiho–Sprang)

Let τ *be any topology finer than the cdp-topology. The inclusion* $\mathrm{Sm}_k \hookrightarrow \mathrm{Var}_k$ *induces an equivalence of topoi*

$$Sh(\mathrm{Sm}_{k,\tau}) \xrightarrow{\sim} Sh(\mathrm{Var}_{k,\tau}).$$

Proof.

- $\mathrm{Sm}_k \hookrightarrow \mathrm{Var}_k$ is fully faithful.
- By Verdier's result it suffices to show that every k-variety has a *cdp*-cover by smooth k-varieties.
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Lemma

*Under **(SR)**, the cdp-topoology on Sm_k is generated by smooth blow-ups.*

Improving the topology on Sm_k

In this part, we need (SR), (ER), (ERB).

- We want to embed the objects $X \in \mathrm{Sm}_k$ into objects $(X, \overline{X}) \in \mathrm{Var}_k^{nc}$.
- We want to do something similar with *cdp*-morphisms and Zariski morphisms.

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Lemma (E–Shiho–Sprang)

Under ***(SR)***, ***(ER)***, *every $X \in \mathrm{Sm}_k$ has an snc-compactification $\overline{X}$, i.e. $(X, \overline{X}) \in \mathrm{Var}_k^{nc}$.*
For fixed X, the category of snc-compactifications is filtered, denoted by $\{(X, \overline{X})/X\}$.

A good smooth blow-up in Sm_k ...

... is a smooth blow-up square which embeds into nc-pairs

$$
\begin{array}{ccc}
Z' & \hookrightarrow & X' \\
\downarrow & & \downarrow p \\
Z & \overset{e}{\hookrightarrow} & X
\end{array}
\qquad\qquad
\begin{array}{ccc}
(Z', \overline{Z}') & \hookrightarrow & (X', \overline{X}') \\
\downarrow & & \downarrow p \\
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\end{array}
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such that all morphisms are strict and p is a blow-up with centre $\overline{Z}$.

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Proposition (E–Shiho–Sprang)

*Assume **(SR)**, **(ER)**, **(ERB)**.*

Every smooth blow-up $\begin{array}{ccc} Z' & \hookrightarrow & X' \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & X \end{array}$ *is good.*

Proof.

- Take an snc-compactification $(X, \overline{X}_1)$. Let $\overline{Z}_1$ be the closure of Z in $\overline{X}_1$.
- By **(ERB)** there is a commutative diagram

$$\begin{array}{ccc} (Z, \overline{Z}) & \hookrightarrow & (X, \overline{X}) \\ \downarrow & & \downarrow \\ (Z, \overline{Z}_1) & \hookrightarrow & (X, \overline{X}_1) \end{array}$$

  such that $(Z, \overline{Z}) \hookrightarrow (X, \overline{X})$ is a strict closed immersion of nc-pairs and the vertical morphisms are strict proper birational.
- By setting $\overline{X}' := \mathrm{Bl}_{\overline{Z}}(\overline{X})$ one obtains the desired diagram.

□

A good smooth Zariski square in Sm_k ...

... is a smooth Zariski square

$$\begin{array}{ccc} Y & \hookrightarrow & V \\ \downarrow & & \downarrow i \\ U & \overset{j}{\hookrightarrow} & X = i(U) \cup j(V) \end{array}$$

which embeds into one of the following

$$\begin{array}{ccc} (Y, \overline{X}) & \hookrightarrow & (V, \overline{X}) \\ \downarrow & & \downarrow \\ (U, \overline{X}) & \hookrightarrow & (X, \overline{X}) \end{array} \qquad \begin{array}{ccc} \varnothing & \longrightarrow & \varnothing \\ \downarrow & & \downarrow \\ (X, \overline{X}) & \longrightarrow & (X, \overline{X}) \end{array} \qquad \begin{array}{ccc} \varnothing & \longrightarrow & (X, \overline{X}) \\ \downarrow & & \downarrow \\ \varnothing & \longrightarrow & (X, \overline{X}) \end{array}$$

For Zariski squares, we only have an embedding result *cdp*-locally (but this is good enough).

Proposition (E–Shiho–Sprang)

Assume ***(SR)***, ***(ER)***. *Given a Zariski square in* Var_k

$$\begin{array}{ccc} Y & \hookrightarrow & V \\ \big\downarrow & & \big\downarrow i \\ U & \overset{j}{\hookrightarrow} & X \end{array}$$

there exists a cdp-hypercovering $X_\bullet \to X$, *along which the pull-back*

$$\begin{array}{ccc} Y_\bullet & \hookrightarrow & V_\bullet \\ \big\downarrow & & \big\downarrow i \\ U_\bullet & \overset{j}{\hookrightarrow} & X_\bullet \end{array}$$

is a good simplicial Zariski square.

Construction for smooth open varieties

In this part, we need (SR), (ER), (ERB), (WF).

- We will use log structures:
  - $k := (\mathrm{Spec}(k), triv)$, $W_n(k) := (\mathrm{Spec}(W_n(k)), triv)$, $W(k) := (\mathrm{Spf}(W(k)), triv)$
  - We consider $(X, \overline{X}) \in \mathrm{Var}_k^{nc/geo}$ as a log scheme $\overline{X}$ with log structure induced by $\overline{X} \backslash X$.

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  - We consider $(X, \overline{X}) \in \mathrm{Var}_k^{nc/geo}$ as a log scheme $\overline{X}$ with log structure induced by $\overline{X} \backslash X$.
- For $X \in \mathrm{Sm}_k$: $W_n\Omega^\bullet_{X/k}$, $W\Omega^\bullet_{X/k}$, which computes crystalline cohomology $R\Gamma_{\mathrm{cris}}(X/W_n(k)) = R\Gamma(X, W_n\Omega^\bullet_{X/k})$.
- For $(X, \overline{X}) \in \mathrm{Var}_k^{nc}$: $W_n\omega^\bullet_{(X,\overline{X})/k} = W_n\Omega^\bullet(\overline{X} \backslash X)_{\overline{X}/k}$, which computes log-crystalline cohomology $R\Gamma_{\mathrm{cris}}((X, \overline{X})/W_n(k)) = R\Gamma(\overline{X}, W_n\omega^\bullet_{(X,\overline{X})/k})$.

For $(X, \overline{X}) \in \mathrm{Var}_k^{nc}$ let $A_n^\bullet(X, \overline{X})$ be an explicit complex functorial in $(X, \overline{X})$ representing $R\Gamma(\overline{X}, W_n\omega^\bullet_{(X,\overline{X}/k)})$.

Proposition (E–Shiho–Sprang)

*Assume **(SR)**, **(ER)**, **(WF)**. For fixed $X \in \mathrm{Sm}_k$ and varying $(X, \overline{X}) \in \mathrm{Var}_k^{nc}$ all $A_n^\bullet(X, \overline{X})$ are quasi-isomorphic.*

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Proof.

- We know that $\{(X,\overline{X})/X\}$ is filtered.
- We only need to show that for a strict proper morphism $(X,\overline{X}) \to (X,\overline{X}')$ the induced morphism $A_n(X,\overline{X}') \xrightarrow{\sim} A_n^\bullet(X,\overline{X})$.
- By weak factorisation we may assume that this is a blow-up with smooth centre.
- We work Zariski-locally, i.e. with affine schemes.
- Then everything lifts and we can compute explicitely.

□

Definition

Assume **(SR)**, **(ER)**, **(WF)**. For $X \in \mathrm{Sm}_k$ define

$$A_n^\bullet(X) := \varinjlim_{(X,\overline{X})} A_n^\bullet(X,\overline{X}), \qquad A^\bullet(X) := R\varprojlim_n A_n^\bullet(X).$$

With this definition $A_n^\bullet(X) \cong A_n^\bullet(X,\overline{X})$ for all $(X,\overline{X}) \in \mathrm{Var}_k^{nc}$.

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Proposition (E–Shiho–Sprang)

*Assume **(SR)**, **(ER)**, **(WF)**, then $A_n^\bullet(X)$ and $A^\bullet(X)$ are functorial in X.*

Proof.

- For $f : X \to Y$ in Sm_k consider the category

$$\{\overline{f}/f\} = \{\overline{f} : (X, \overline{X}) \to (Y, \overline{Y}) \mid (X, \overline{X}), (Y, \overline{Y}) \in \mathrm{Var}_k^{nc}\}.$$

- Show this category is non-empty and filtered.
- There are projections, where p_2 is surjective

$$\begin{array}{ccc} & \{\overline{f}/f\} & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ \{(X,\overline{X})/X\} & & \{(Y,\overline{Y})/Y\} \end{array}$$

- Any extension $\overline{f}$ of f induces a natural morphism $A_n^\bullet(p_2(\overline{f})) \to A_n^\bullet(p_1(\overline{f}))$.
- We obtain a zig-zag $\varinjlim_{\overline{Y}} A_n^\bullet(Y, \overline{Y}) \xleftarrow{\sim} \varinjlim_{\overline{f}} A_n^\bullet(p_2(\overline{f})) \to \varinjlim_{\overline{f}} A_n^\bullet(p_1(\overline{f})) \to \varinjlim_{\overline{X}} A_n^\bullet(X, \overline{X})$

□

- We may regard $A_n^\bullet$ as a complex of presheaves on Sm_k.
- Sheafify it with respect to the *cdp*- and *rh*-topology.

Definition

Define the following complexes of sheaves on Sm_k

$$a_{cdp}^* A_n^\bullet \qquad\qquad a_{rh}^* A_n^\bullet$$

$$a_{cdp}^* A^\bullet := R\varprojlim a_{cdp}^* A_n^\bullet \qquad\qquad a_{rh}^* A^\bullet := R\varprojlim a_{rh}^* A_n^\bullet$$

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Proposition (E–Shiho–Sprang)

Assume ***(SR)***, ***(ER)***, ***(ERB)***, ***(WF)***. *For every* $X \in \mathrm{Sm}_k$ *the natural morphisms*

$$A^\bullet_n(X) \to R\Gamma_{cdp}(X, a^*_{cdp}A^\bullet_n) \to R\Gamma_{rh}(X, a^*_{rh}A^\bullet_n)$$

are quasi-isomorphisms.

Proof.

- By work of Cortinas–Haesemayer–Schlichting–Weibel it suffices to show that A_n satisfies the Mayer–Wietoris property: For a smooth blow-up square $\begin{array}{ccc} Z' & \hookrightarrow & X' \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & X \end{array}$

  the induced diagram $\begin{array}{ccc} A_n^\bullet(Z')] & \longleftarrow & A_n^\bullet(X') \\ \uparrow & & \uparrow \\ A_n^\bullet(Z) & \longleftarrow & A_n^\bullet(X) \end{array}$ is homoropy co-cartesian.
- To show this, we take a good compactification of the square and work Zariski locally.
- Then compute explicitely.
- Similarly for Zariski square, except that we use the previous result to work *cdp*-locally.

□

Extension to k-varieties

In this part, we need (SR), (ER), (ERB), (WF).

- We now want to use the equivalence of topoi $Sh(\mathrm{Sm}_{k,\tau}) \xrightarrow{\sim} Sh(\mathrm{Var}_{k,\tau})$ to extend the construction to Var_k.

Definition

By the above equivalence of topoi

$$a^*_{cdp}A^\bullet_n \qquad\qquad a^*_{rh}A^\bullet_n$$

$$a^*_{cdp}A^\bullet := R\varprojlim a^*_{cdp}A^\bullet_n \qquad a^*_{rh}A^\bullet := R\varprojlim a^*_{rh}A^\bullet_n$$

define (complexes of) sheaves on Var_k.

We have a similar descent result as before:

Proposition (E–Shiho–Sprang)

Assume ***(SR)***, ***(ER)***, ***(ERB)***, ***(WF)*** *Then for any* $X \in \mathrm{Var}_k$

$$R\Gamma_{cdp}(X, a_{cdp}^{*}A_n^{\bullet}) \xrightarrow{\sim} R\Gamma_{rh}(X, a_{rh}^{*}A_n^{\bullet}).$$

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Proof.

Choose a *cdp*-hypercovering $X_\bullet \to X$ with $X_i \in \mathrm{Sm}_k$. Then there is a commutative diagram

$$\begin{array}{ccc} R\Gamma_{cdp}(X, a_{cdp}^* A_n^\bullet) & \longrightarrow & R\Gamma_{rh}(X, a_{rh}^* A_n^\bullet) \\ \downarrow \wr & & \downarrow \wr \\ R\Gamma_{cdp}(X_\bullet, a_{cdp}^* A_n^\bullet) & \xrightarrow{\sim} & R\Gamma_{rh}(X_\bullet, a_{rh}^* A_n^\bullet) \end{array}$$

where the vertical maps are quasi-isomorphisms because of *cdh*-descent, and the lower horizontal map is a quasi-isomorphism because all maps $R\Gamma_{cdp}(X_i, a_{cdp}^* A_n^\bullet) \xrightarrow{\sim} R\Gamma_{rh}(X_i, a_{rh}^* A_n^\bullet)$. □

Some properties of $H^*_{rh}(X, a^*_{rh}A^\bullet)$ and $H^*_{cdp}(X, a^*_{cdp}A^\bullet)$:

- For $(X, \overline{X}) \in \mathrm{Var}_k^{nc}$:
  $H^*_{cris}((X, \overline{X})/W(k)) \cong H^*_{cdp}(X, a^*_{cdp}A^\bullet) \cong H^*_{rh}(X, a^*_{rh}A^\bullet)$.
- The cohomology groups $H^n_{rh}(X, a^*_{rh}A^\bullet)$ are finitely generated.
- $H^n_{rh}(X, a^*_{rh}A^\bullet) = 0$ for $n < 0$, $n > 2\dim(X)$.
- There is a canonical quasi-isomorphism
  $H^*_{rig}(X/K) \cong H^*_{rh}(X, a^*_{rh}A^\bullet) \otimes \mathbb{Q}$.
- Satisfies the Künneth formula.
- And hence homotopy invariance.
- Has Chern classes compatible with crystalline and rigid Chern classes.

Alterations instead of resolutions

- Use de Jong's alteration theorem.
- **But:** this means that we allow finite extensions, which is a problem in positive characteristic!
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Let $X \in \mathrm{Var}_k$.

- By Nagata, we obtain $(X, \overline{X}) \in \mathrm{Var}_k^{geo}$.
- By Nakkajima, we obtain a split proper hypercovering $(X_\bullet, \overline{X}_\bullet) \to (X, \overline{X})$ by nc-pairs.
- It is known that $H^*_{rig}(X/K) \cong H^*_{cris}((X_\bullet, \overline{X}_\bullet)/W(k)) \otimes \mathbb{Q}$.

Question

Is $H^*_{cris}((X_\bullet, \overline{X}_\bullet)/W(k))$ independent of the choice of hypercovering?

No – not in general:

- Let $X/\mathbb{F}_p$ be an elliptic curve (then $\overline{X} = X$), $F : X \to X$ the absolute Frobenius.
- $X'_\bullet \to X$ the associated Čech hypercover, $X_\bullet := (X'_\bullet)_{red}$.
- Each X_i equals X, $\pi : X_\bullet \to X$ is a split proper hypercovering.
- Induced maps
  $F^* : H^1_{cris}(X/W(k)) \xrightarrow{\pi^*} H^1_{cris}(X_\bullet/W(k)) \xrightarrow{H} {}^1_{cris}\,(X/W(k))$
  where the second map is the edge map of the spectral sequence
  $E_1^{ij} = H^i_{cris}(X_i/W(k)) \Rightarrow H^1_{cris}(X_\bullet/W(k))$.
- Since $H^1_{cris}(X/W(k))$ has non-trivial slope part F^* is not an isomorphism.
- π^* also not an isomorphism.

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- Each X_i equals X, $\pi : X_\bullet \to X$ is a split proper hypercovering.
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  $F^* : H^1_{cris}(X/W(k)) \xrightarrow{\pi^*} H^1_{cris}(X_\bullet/W(k)) \xrightarrow{H}{}^{1}_{cris} (X/W(k))$
  where the second map is the edge map of the spectral sequence
  $E_1^{ij} = H^i_{cris}(X_i/W(k)) \Rightarrow H^1_{cris}(X_\bullet/W(k))$.
- Since $H^1_{cris}(X/W(k))$ has non-trivial slope part F^* is not an isomorphism.
- π^* also not an isomorphism.

We have to restrict to **generically étale** hypercoverings (not a problem).

Low cohomological degrees

Here we do have positive results!

- $H^0_{cris}((X_\bullet, \overline{X}_\bullet)/W(k))$: It is well-known that this is independent of the choice of hypercovering.
- $H^0_{cris}((X_\bullet, \overline{X}_\bullet)/W(k))$ the independence was shown
  - by Andreatta–Barbieri-Viale for $p \geqslant 3$.
  - by E–Shiho–Sprang for $p \geqslant 2$.
- This is **not** true for higher cohomological degrees.

Counterexamples for higher cohomological degrees

- Let $\overline{X} = \mathbb{P}^1_k$, x the coordinate of $\mathbb{A}^1_k \subset \mathbb{P}^1_k$.
  For $r \geqslant 1$: $a_1, \ldots a_r \in k$ distinct, $n_1, \ldots, n_r$ prime to p.

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- Let $f : \overline{X}_0 \to \overline{X}$ be the morphism induced by the field extension

$$k(\overline{X}) = k(x) \subseteq k(x)[y]/(y^p - y - \frac{x^{\sum_{i=1}^r n_i}}{\prod_{i=1}^r (x - a_i)^{n_i}}) =: k(\overline{X}_0).$$

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- Let $\overline{X}_i = (\overline{X}_0 \times_{\overline{X}} \cdots \times \overline{X}_0)^{norm} = \coprod^{G^i} \overline{X}_0$.
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- Then $H^2_{cris}((X, \overline{X})/W(k)) \to H^2_{cris}((X_\bullet, \overline{X}_\bullet)/W(k))$ is not an isomorphism.

Thank you very much for your attention!