Integral *p*-adic cohomology theories for open and singular varieties

joint with A. Shiho (Tokyo) and J. Sprang (Duisburg-Essen)

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Dr. Veronika ERTL-BLEIMHOFER



Universität Regensburg

Integral *p*-adic cohomology theories

The search for a good integral *p*-adic cohomology

Construction under resolution of singularities Some useful notions Hypothesis that we need Topologies on Var_k and Sm_k Improving the topology on Sm_k Construction for smooth open varieties Extension to *k*-varieties

What can we do without resolution of singularities? Alterations instead of resolutions Low cohomological degrees Counterexamples for higher cohomological degrees

- Weil conjectures as a starting point
 - concerns the Zeta function of a variety in positive characteristic p
 - notion of Weil cohomology
 - accomplished using ℓ -adic cohomology ($\ell \neq p$)
 - desire to fill the gap for $\ell = p$

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 - k perfect field of characteristic p > 0
 - W(k) ring of Witt vectors of k
 - K = Frac(W(k)) fraction field of W(k)

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Candidates:

- H^{*}_{cris}(X/W(k)), Grothendieck/Berthelot Finitely generated over W(k) only for proper smooth X/k!
- H^{*}_{rig}(X/K), Monsky–Washnitzer (local), Berthelot (global) Is finitely generated over K, but has rational coefficients!
- H*(X, W[†]Ω[•]), Davis–Langer–Zink
 Compares rationally to rigid cohomology, but not finitely generated
 over W(k) even modulo torsion! (Cunterexample: E–Shiho)

Question

Under which conditions can we expect a "good" integral p-adic cohomology theory for open / singular varieties?

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Good means ...

- The cohomology groups H^{*}_{good}(X) are finitely generate W(k)-modules for all X ∈ Var_k.
- ► There is a comparison isomorphism with (log) crystalline cohomology for X ∈ Var_k (log) smooth proper.
- ► There is a rational comparison isomorphism with rigid cohomology for all X ∈ Var_k.

Some useful notions

- ► Geometric pair (X, X): open immersion X → X in Var_k with dense image, X proper Var^{geo}_k
- ► Normal crossing pair (X, X): geometric pair, such that X \X is a simple normal crossing divisor Var^{nc}_k
- ▶ Morphisms of geometric pairs $f : (X_1, \overline{X}_1) \to (X_2, \overline{X}_2)$: a morphism $f : \overline{X}_1 \to \overline{X}_2$ in Var_k such that $f(X_1) \subset X_2$

• strict: if
$$f^{-1}(X_2) = X_1$$

• has property **P**: if $f : \overline{X}_1 \to \overline{X}_2$ has property P

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Conceptually ...

... normal crossing pairs are to geometric pairs, what smooth varieties are to varieties.

Hypothesis that we need

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Strong resolutions of singularities (SR)

- For all $X \in Var_k$ there exists a proper birational morphism $f : X' \to X$, such that X' is smooth, and f is an iso on X_{sm} .
- For all proper birational morphisms f : X' → X in Sm_k there is a sequence of blow-ups along smooth centres

$$X_n \to X_{n-1} \to \cdots \to X$$

that factors through f.

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Embedded resolutions of singularities (EB)

For all $(X,\overline{X}) \in \operatorname{Var}_k^{geo}$ with \overline{X} smooth, there is a proper birational morphism $f: (X,\overline{X}') \to (X,\overline{X})$ such that $(X,\overline{X}') \in \operatorname{Var}_k^{nc}$.

Weak factorisation (WF)

For every strict proper birational morphism $(X, \overline{X}') \to (X, \overline{X})$ in $\operatorname{Var}_k^{nc}$ which is an iso on X, there exists a weak factorisation.

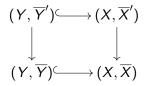
• A weak factorisation of a proper birational morphism $f: (X_1, \overline{X}_1) \to (X_2, \overline{X}_2)$ which is an iso on X_2 is a sequence

$$(X_1,\overline{X}_1) = (V_0,\overline{V}_0^{f_1}) - - + \ldots - + (V_\ell,\overline{V}_\ell) = (X_2,\overline{X}_2)$$

where f_i is rational, $f_{\ell} \circ \ldots \circ f_1 = f$, each composition $f_i \circ \ldots \circ f_1$ is a morphism and induced an iso on X_2 , and for each i either f_i or f_i^{-1} is a blow-up along a smooth center Z_i disjoint from X_2 which has normal crossing with $\overline{V}_i \setminus V_i$ (or $\overline{V}_{i-1} \setminus V_{i-1}$.

Embedded resolutions with boundaries (ERB)

For all strict closed immersions of geometric pairs $(Y, \overline{Y}) \rightarrow (X, \overline{X})$ such that Y is smooth and $(X, \overline{X}) \in \operatorname{Var}_{k}^{nc}$, there is a commutative diagram



where the horizontal maps are strict closed immersions and the vertical maps are proper birational morphisms and isos on Y, X.

Topologies on Var_k and Sm_k

In this part, we only use property (SR).

To make things smooth, we want to consider the topology "generated by blow-ups".

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The *cdp*-topology on $Var_k \ldots$

... is the topology generated by completely decomposed proper morphisms $p: Y \to X$.

(Completely decomposed means that for every $x \in X$ there is

 $y \in p^{-1}(x) \subset Y$, such that for the residue fields $\kappa(x) \xrightarrow{\sim} \kappa(y)$.)

Lemma (Suslin–Voevodsky)

The cdp-topoology on Var_k is generated by blow-ups.

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Lemma (Suslin–Voevodsky)

The cdp-topoology on Var_k is generated by blow-ups.

The *rh*-topology on $Var_k \ldots$

... is the topology generated by *cdp*-morphisms and Zariski morphisms.

One can restrict these topologies to Sm_k

Proposition (E–Shiho–Sprang)

Let τ be any topology finer than the cdp-topology. The inclusion $Sm_k \hookrightarrow Var_k$ induces an equivalence of topoi

$$Sh(Sm_{k,\tau}) \xrightarrow{\sim} Sh(Var_{k,\tau}).$$

Proof.

- Sm_k \hookrightarrow Var_k is fully faithful.
- By Verdier's result it suffices to show that every k-variety has a cdp-cover by smooth k-varieties.
- This follows with (SR).

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Lemma

Under (SR), the cdp-topoology on Sm_k is generated by smooth blow-ups.

Improving the topology on Sm_k

In this part, we need (SR), (ER), (ERB).

- ▶ We want to embed the objects $X \in Sm_k$ into objects $(X, \overline{X}) \in Var_k^{nc}$.
- We want to do something similar with *cdp*-morphisms and Zariski morphisms.

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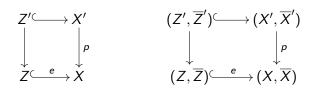
- ▶ We want to embed the objects $X \in Sm_k$ into objects $(X, \overline{X}) \in Var_k^{nc}$.
- We want to do something similar with *cdp*-morphisms and Zariski morphisms.

Lemma (E-Shiho-Sprang)

Under **(SR)**, **(ER)**, every $X \in Sm_k$ has an snc-compactification \overline{X} , *i.e.* $(X, \overline{X}) \in Var_k^{nc}$. For fixed X, the category of snc-compactifications is filtered, denoted by $\{(X, \overline{X})/X\}$.

A good smooth blow-up in $Sm_k \ldots$

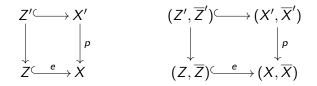
... is a smooth blow-up square which embeds into nc-pairs



such that all morphisms are strict and p is a blow-up with centre \overline{Z} .

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Proposition (E–Shiho–Sprang)

Assume (SR), (ER), (ERB). Every smooth blow-up $Z' \longrightarrow X'$ is good.

Proof.

- Take an snc-compactification (X, \overline{X}_1) . Let \overline{Z}_1 be the closure of Z in \overline{X}_1 .
- By (ERB) there is a commutative diagram

such that $(Z, \overline{Z}) \hookrightarrow (X, \overline{X})$ is a strict closed immersion of nc-pairs and the vertical morphisms are strict proper birational.

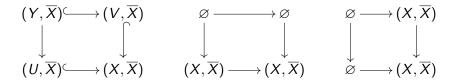
• By setting $\overline{X}' := Bl_{\overline{Z}}(\overline{X})$ one obtains the desired diagram.

A good smooth Zariski square in $Sm_k \ldots$

... is a smooth Zariski square

$$Y \xrightarrow{} V \\ \downarrow \qquad \qquad \downarrow^{i} \\ U \xrightarrow{j} X = i(U) \cup j(V)$$

which embeds into one of the following



For Zariski squares, we only have an embedding result *cdp*-locally (but this is good enough).

Proposition (E-Shiho-Sprang)

Assume (SR), (ER). Given a Zariski square in Var_k



there exists a cdp-hypercovering $X_{\bullet} \rightarrow X$, along which the pull-back



is a good simplicial Zariski square.

Construction for smooth open varieties

In this part, we need (SR), (ER), (ERB), (WF).

- We will use log structures:
 - k := (Spec(k), triv), W_n(k) := (Spec(W_n(k)), triv), W(k) := (Spf(W(k)), triv)
 - We consider $(X, \overline{X}) \in \operatorname{Var}_k^{nc/geo}$ as a log scheme \overline{X} with log structure induced by $\overline{X} \setminus X$.

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For
$$X \in \text{Sm}_k$$
: $W_n \Omega^{\bullet}_{X/k}$, $W \Omega^{\bullet}_{X/k}$,
which computes crystalline cohomology
 $R\Gamma_{\text{cris}}(X/W_n(k)) = R\Gamma(X, W_n \Omega^{\bullet}_{X/k})$.

► For $(X, \overline{X}) \in \operatorname{Var}_{k}^{nc}$: $W_{n}\omega_{(X,\overline{X})/k}^{\bullet} = W_{n}\Omega^{\bullet}(\overline{X}\setminus X)_{\overline{X}/k}$, which computes log-crystalline cohomology $R\Gamma_{\operatorname{cris}}((X,\overline{X})/W_{n}(k)) = R\Gamma(\overline{X}, W_{n}\omega_{(X,\overline{X})/k}^{\bullet}).$ For $(X, \overline{X}) \in \operatorname{Var}_{k}^{nc}$ let $A_{n}^{\bullet}(X, \overline{X})$ be an explicit complex functorial in (X, \overline{X}) representing $R\Gamma(\overline{X}, W_{n}\omega_{(X,\overline{X}/k)}^{\bullet})$.

Proposition (E-Shiho-Sprang)

Assume **(SR)**, **(ER)**, **(WF)**. For fixed $X \in \text{Sm}_k$ and varying $(X, \overline{X}) \in \text{Var}_k^{nc}$ all $A_n^{\bullet}(X, \overline{X})$ are quasi-isomorphic.

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Proof.

- We know that $\{(X, \overline{X})/X\}$ is filtered.
- ▶ We only need to show that for a strict proper morphism $(X, \overline{X}) \rightarrow (X, \overline{X}')$ the induced morphism $A_n(X, \overline{X}') \xrightarrow{\sim} A_n^{\bullet}(X, \overline{X})$.
- By weak factorisation we may assume that this is a blow-up with smooth centre.
- We work Zariski-locally, i.e. with affine schemes.
- Then everything lifts and we can compute explicitely.

Definition

Assume (SR), (ER), (WF). For $X \in Sm_k$ define

$$A_n^{\bullet}(X) := \varinjlim_{(X,\overline{X})} A_n^{\bullet}(X,\overline{X}), \qquad A^{\bullet}(X) := R \varprojlim_n A_n^{\bullet}(X).$$

With this definition $A_n^{\bullet}(X) \cong A_n^{\bullet}(X, \overline{X})$ for all $(X, \overline{X}) \in \operatorname{Var}_k^{nc}$.

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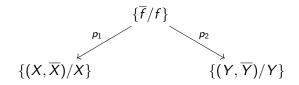
Assume (SR), (ER), (WF), then $A_n^{\bullet}(X)$ and $A^{\bullet}(X)$ are functorial in X.

Proof.

• For $f: X \to Y$ in Sm_k consider the category

$$\{\overline{f}/f\} = \{\overline{f}: (X,\overline{X}) \to (Y,\overline{Y}) \mid (X,\overline{X}), (Y,\overline{Y}) \in \operatorname{Var}_k^{nc}\}.$$

- Show this category is non-empty and filtered.
- There are projections, where p₂ is surjective



- Any extension \overline{f} of f induces a natural morphism $A_n^{\bullet}(p_2(\overline{f})) \to A_n^{\bullet}(p_1(\overline{f})).$
- ► We obtain a zig-zag $\varinjlim_{\overline{Y}} A^{\bullet}_n(Y, \overline{Y}) \xleftarrow{\sim} \varinjlim_{\overline{f}} A^{\bullet}_n(p_2(\overline{f})) \to \varinjlim_{\overline{f}} A^{\bullet}_n(p_1(\overline{f})) \to \varinjlim_{\overline{X}} A^{\bullet}_n(X, \overline{X})$

- We may regard A_n^{\bullet} as a complex of presheaves on Sm_k .
- Sheafify it with respect to the *cdp* and *rh*-topology.

Definition

Define the following complexes of sheaves on Sm_k

$$a_{cdp}^*A^{\bullet} = R \varprojlim a_{cdp}^*A^{\bullet} \qquad \qquad a_{rh}^*A^{\bullet} = R \varprojlim a_{rh}^*A^{\bullet}$$

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Proposition (E-Shiho-Sprang)

Assume (SR), (ER), (ERB), (WF). For every $X \in Sm_k$ the natural morphisms

$$A^{\bullet}_{n}(X) \to R\Gamma_{cdp}(X, a^{*}_{cdp}A^{\bullet}_{n}) \to R\Gamma_{rh}(X, a^{*}_{rh}A^{\bullet}_{n})$$

are quasi-isomorphisms.

Proof.

- By work of Cortinas–Haesemayer–Schlichting–Weibel it suffices to show that A_n satisfies the Mayer–Wietoris property: For a smooth blow-up square $Z' \longrightarrow X'$ the induced diagram $A_n^{\bullet}(Z')$] $\leftarrow A_n^{\bullet}(X')$ is homoropy co-cartesian. $A_n^{\bullet}(Z) \longleftarrow A_n^{\bullet}(X)$
- To show this, we take a good compactification of the square and work Zariski locally.
- Then compute explicitely.
- Similarly for Zariski square, except that we use the previous result to work *cdp*-locally.

Extension to k-varieties

In this part, we need (SR), (ER), (ERB), (WF).

▶ We now want to use the equivalence of topol $Sh(Sm_{k,\tau}) \xrightarrow{\sim} Sh(Var_{k,\tau})$ to extend the construction to Var_k .

Definition

By the above equivalence of topoi

$$a_{cdp}^*A^{\bullet} \qquad \qquad a_{rh}^*A^{\bullet}_n \qquad \qquad a_{rh}^*A^{\bullet}_n \\ a_{cdp}^*A^{\bullet} := R \varprojlim a_{cdp}^*A^{\bullet}_n \qquad \qquad a_{rh}^*A^{\bullet} := R \varprojlim a_{rh}^*A^{\bullet}_n$$

define (complexes of) sheaves on Var_k .

We have a similar descent result as before:

Proposition (E–Shiho–Sprang)

Assume (SR), (ER), (ERB), (WF) Then for any $X \in Var_k$

 $R\Gamma_{cdp}(X, a^*_{cdp}A^{\bullet}_n) \xrightarrow{\sim} R\Gamma_{rh}(X, a^*_{rh}A^{\bullet}_n).$

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Proof.

Choose a *cdp*-hypercovering $X_{\bullet} \to X$ with $X_i \in \text{Sm}_k$. Then there is a commutative diagram

where the vertical maps are quasi-isomorphisms because of *cdh*-descent, and the lower horizontal map is a quasi-isomorphism because all maps $R\Gamma_{cdp}(X_i, a^*_{cdp}A^{\bullet}_n) \xrightarrow{\sim} R\Gamma_{rh}(X_i, a^*_{rh}A^{\bullet}_n).$ Some properties of $H^*_{rh}(X, a^*_{rh}A^{\bullet})$ and $H^*_{cdp}(X, a^*_{cdp}A^{\bullet})$:

- For $(X,\overline{X}) \in \operatorname{Var}_k^{nc}$: $H^*_{cris}((X,\overline{X})/W(k)) \cong H^*_{cdp}(X, a^*_{cdp}A^{\bullet}) \cong H^*_{rh}(X, a^*_{rh}A^{\bullet}).$
- The cohomology groups $H_{rh}^n(X, a_{rh}^*A^{\bullet})$ are finitely generated.

•
$$H^n_{rh}(X, a^*_{rh}A^{\bullet}) = 0$$
 for $n < 0, n > 2 \dim(X)$.

- ► There is a canonical quasi-isomorphism $H^*_{rig}(X/K) \cong H^*_{rh}(X, a^*_{rh}A^{\bullet}) \otimes \mathbb{Q}.$
- Satisfies the Künneth formula.
- And hence homotopy invariance.
- Has Chern classes compatible with crystalline and rigid Chern classes.

Alterations instead of resolutions

- Use de Jong's alteration theorem.
- But: this means that we allow finite extensions, which is a problem in positive characteristic!
- The topology generated by alterations is the proper topology.

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Let $X \in Var_k$.

- ▶ By Nagata, we obtain $(X, \overline{X}) \in \operatorname{Var}_{k}^{geo}$.
- ▶ By Nakkajima, we obtain a split proper hypercovering $(X_{\bullet}, \overline{X}_{\bullet}) \rightarrow (X, \overline{X})$ by nc-pairs.
- ► It is known that $H^*_{rig}(X/K) \cong H^*_{cris}((X_{\bullet}, \overline{X}_{\bullet})/W(k)) \otimes \mathbb{Q}$.

Question

Is $H^*_{cris}((X_{\bullet}, \overline{X}_{\bullet})/W(k))$ independent of the choice of hypercovering?

No – not in general:

- ▶ Let X/\mathbb{F}_p be an elliptic curve (then $\overline{X} = X$), $F : X \to X$ the absolute Frobenius.
- ▶ $X'_{\bullet} \to X$ the associated Čech hypercover, $X_{\bullet} := (X'_{\bullet})_{red}$.
- ▶ Each X_i equals $X, \pi : X_{\bullet} \to X$ is a split proper hypercovering.
- Induced maps
 - $F^*: H^1_{cris}(X/W(k)) \xrightarrow{\pi^*} H^1_{cris}(X_{\bullet}/W(k)) \xrightarrow{H^1_{cris}} (X/W(k))$ where the second map is the edge map of the spectral sequence $E^{ij}_1 = H^i_{cris}(X_i/W(k)) \Rightarrow H^1_{cris}(X_{\bullet}/W(k)).$
- Since H¹_{cris}(X/W(k)) has non-trivial slope part F* is not an isomorphism.
- π^* also not an isomorphism.

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We have to restrict to generically étale hypercoverings (not a problem).

Here we do have positive results!

- → H⁰_{cris}((X_•, X̄_•)/W(k)): It is well-known that this is independent of the choice of hypercovering.
- $H^0_{cris}((X_{\bullet}, \overline{X}_{\bullet})/W(k))$ the independence was shown
 - by Andreatta–Barbieri-Viale for $p \ge 3$.
 - by E–Shiho–Sprang for $p \ge 2$.
- This is not true for higher cohomological degrees.

▶ Let $\overline{X} = \mathbb{P}^1_k$, x the coordinate of $\mathbb{A}^1_k \subset \mathbb{P}^1_k$. For $r \ge 1$: $a_1, \ldots a_r \in k$ distinct, n_1, \ldots, n_r prime to p.

Let X = P¹_k, x the coordinate of A¹_k ⊂ P¹_k. For r ≥ 1: a₁,... a_r ∈ k distinct, n₁,..., n_r prime to p.
Let f : X
₀ → X be the morphism induced by the field extension k(X) = k(x) ⊆ k(x)[y]/(y^p - y - (x ∑^r_{i=1} n_i)/n_i) =: k(X
₀).

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$$k(\overline{X}) = k(x) \subseteq k(x)[y]/(y^p - y - \frac{x^{\sum_{i=1}^r n_i}}{\prod_{i=1}^r (x - a_i)^{n_i}}) =: k(\overline{X}_0).$$

Finite flat morphism of degree p between proper smooth curves such that k(X
0)/k(X) is a Galois extension with Galois group G = ⟨g⟩ ≅ Z/pZ.

- ▶ Let $\overline{X} = \mathbb{P}^1_k$, x the coordinate of $\mathbb{A}^1_k \subset \mathbb{P}^1_k$. For $r \ge 1$: $a_1, \ldots a_r \in k$ distinct, n_1, \ldots, n_r prime to p.
- ► Let $f : \overline{X}_0 \to \overline{X}$ be the morphism induced by the field extension $k(\overline{X}) = k(x) \subseteq k(x)[y]/(y^p - y - \frac{x^{\sum_{i=1}^r n_i}}{\prod_{i=1}^r (x - a_i)^{n_i}}) =: k(\overline{X}_0).$
- Finite flat morphism of degree p between proper smooth curves such that k(X
 0)/k(X) is a Galois extension with Galois group G = ⟨g⟩ ≅ Z/pZ.
- Let P_i = {x = a_i} a closed point of X̄. Ramification locus of f is D = ∪ P_i.

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- ▶ Let $\overline{X}_i = (\overline{X}_0 \times_{\overline{X}} \cdots \times \overline{X}_0)^{norm} = \coprod^{G^i} \overline{X}_0.$ Simplicial scheme $\overline{X}_{\bullet} \to X.$

- ▶ Let $\overline{X} = \mathbb{P}_k^1$, x the coordinate of $\mathbb{A}_k^1 \subset \mathbb{P}_k^1$. For $r \ge 1$: $a_1, \ldots a_r \in k$ distinct, n_1, \ldots, n_r prime to p.
- ► Let $f : \overline{X}_0 \to \overline{X}$ be the morphism induced by the field extension $k(\overline{X}) = k(x) \subseteq k(x)[y]/(y^p - y - \frac{x^{\sum_{i=1}^r n_i}}{\prod_{i=1}^r (x - a_i)^{n_i}}) =: k(\overline{X}_0).$
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- ▶ Let $X := \overline{X} \setminus D$ and $X_{\bullet} = X \times_{\overline{X}} \overline{X}_{\bullet}$. Split proper generically étale hypercovering $(X_{\bullet}, \overline{X}_{\bullet}) \to (X, \overline{X})$.

- ▶ Let $\overline{X} = \mathbb{P}_k^1$, x the coordinate of $\mathbb{A}_k^1 \subset \mathbb{P}_k^1$. For $r \ge 1$: $a_1, \ldots, a_r \in k$ distinct, n_1, \ldots, n_r prime to p.
- ► Let $f : \overline{X}_0 \to \overline{X}$ be the morphism induced by the field extension $k(\overline{X}) = k(x) \subseteq k(x)[y]/(y^p - y - \frac{x^{\sum_{i=1}^r n_i}}{\prod_{i=1}^r (x - a_i)^{n_i}}) =: k(\overline{X}_0).$
- Finite flat morphism of degree p between proper smooth curves such that k(X
 0)/k(X) is a Galois extension with Galois group G = ⟨g⟩ ≅ Z/pZ.
- Let $P_i = \{x = a_i\}$ a closed point of \overline{X} . Ramification locus of f is $D = \bigcup P_i$.
- Let $\overline{X}_i = (\overline{X}_0 \times_{\overline{X}} \cdots \times \overline{X}_0)^{norm} = \coprod^{G^i} \overline{X}_0.$ Simplicial scheme $\overline{X}_{\bullet} \to X.$
- ▶ Let $X := \overline{X} \setminus D$ and $X_{\bullet} = X \times_{\overline{X}} \overline{X}_{\bullet}$. Split proper generically étale hypercovering $(X_{\bullet}, \overline{X}_{\bullet}) \to (X, \overline{X})$.
- ▶ Then $H^2_{cris}((X,\overline{X})/W(k)) \rightarrow H^2_{cris}((X_{\bullet},\overline{X}_{\bullet})/W(k))$ is not an isomorphism.

Thank you very much for your attention!