POINCARÉ DUALITY IN LOG RIGID COHOMOLOGY TALK AT THE CONFERENCE "AROUND P-ADIC COHOMOLOGIES" ON THE OCCASION OF BRUNO CHIARELLOTTO'S 60TH BIRTHDAY

by

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Abstract. — Poincaré duality is an important feature of a "good" cohomology that plays a role in many applications. In joint work with Kazuki Yamada (Keio University) we investigate it in the context of log rigid cohomology. I will explain our construction of log rigid cohomology with compact support for several types of coefficients. In particular, I will explain its compatibility with the structures of Hyodo–Kato theory. This approach has the advantage, that the constructions are explicit yet versatile, and hence suitable for computations. (Joint work with Kazuki Yamada, Keio University.)

Contents

Thank you for the invitation, it is a pleasure to be here at this conference in honour of Bruno Chiarellotto. Today I want to talk ybout some constructions concerning rigid cohomology, more precisely log rigid cohomology obtained together with my collaborator Kazuki Yamada from Keio University, that were in some sense inspired by work of Bruno and by discussions with him (eg. [**4**]).

The Weil conjectures can be seen as a starting point for the study of p-adic cohomology theories. Weil has suggested to use a suitable cohomology theory to solve these conjectures for proper and smooth varieties over a field k of characteristic p. For $\ell \neq p$, this has long been solved by Grothendieck's school using ℓ-adic cohomology. One motivation for the research of p-adic cohomology theories is the desire to fill the gap for $\ell = p$.

A first candidate was Berthelot's crystalline cohomology $H^*_{\mathrm{cris}}(X/W)$ [**1**] after a suggestion by Grothendieck. A drawback of crystalline cohomology is that it works well only for proper and smooth schemes – for singular or non-proper schemes, the crystalline cohomology groups are not necessarily finitely generated over W. In this case rigid cohomology $H^i_{\mathrm{rig}}(X/K)$ introduced by Berthelot [**2**], [**3**] has become an important tool.

Key words and phrases. — Logarithmic geometry, rigid cohomology, Poincaré duality.

However, from the point of view of the study of p-adic L-functions, it doesn't seem to be the correct cohomology to consider for non-smooth schemes. One is lead to consider certain logarithmic cohomology theories instead, in particular the so called Hyodo–Kato theory [**6**].

Thus an important question to ask is, what is the difference of these two theories.

0.1. Question. — *Can we describe the difference and relation between rigid cohomology and Hyodo–Kato cohomology in mathematical term?*

0.2. Notation. — I will use the following notation:

V	–	complete discrete valuation ring of mixed characteristic $(0,p)$;
$\mathfrak{m}$	–	its maximal ideal;
K	–	its fraction field;
$\overline{K}$	–	an algebraic closure of K;
G_K	–	the absolute Galois group of K;
k	–	its residue field, which is perfect;
$W := W(k)$	–	the ring of Witt vectors of k;
F	–	the fraction field of W.

For a scheme X/V we denote by

X_n	–	for $n \in \mathbb{N}$, the reduction of X modulo p^n;
X_0	–	its special fibre;
X_K	–	its generic fibre.

We consider the following situation:

X – a flat projective regular V-scheme of relative dimension d.

In this situation Flach and Morin in [**5**] conjecture the following:

0.3. Conjecture. — *There is an exact triangle in the derived category of φ-modules*

(1)

$$R\Gamma_{\mathrm{rig}}(X_0/F) \xrightarrow{\mathrm{sp}} \left[R\Gamma^B_{\mathrm{HK},\mathrm{h}}(X_{\overline{K}})^{G_K} \xrightarrow{N} R\Gamma^B_{\mathrm{HK},\mathrm{h}}(X_{\overline{K}})(-1)^{G_K}\right] \xrightarrow{\mathrm{sp}'} R\Gamma_{\mathrm{rig}}(X_0/F)^*(-d-1)[-2d-1] \to$$

where sp *is the specialisation map and* sp' *is the composition of the Poincaré duality morphism*

$$R\Gamma^B_{\mathrm{HK},\mathrm{h}}(X_{\overline{K}})(-1) \cong R\Gamma^B_{\mathrm{HK},\mathrm{h}}(X_{\overline{K}})^*(-d-1)[-2d] \to$$

and the dual sp^*. *This triangle can be interpreted as a localisation triangle.*

In light that we had recently developed a rigid analytic version of Hyodo–Kato cohomology, we wanted to look at this problem from a purely rigid analytic perspective. We realised however that we would need a compactly supported version of log rigid cohomology and Poincaré duality, in particular to interpret the maps in the above triangle. This will be today's topic.

1. Log rigid cohomology

I first want to recall briefly our definition of log rigid cohomology and in particular rigid Hyodo–Kato cohomology.

We have a base log scheme T (of Zariski type and of finite type) over $\mathbb{F}_p$ together with a "lift" to a weak formal log scheme (of Zariski type) $\mathcal{T}$ over $\mathbb{Z}_p$ (more precisely with a homeomorphic exact closed immersion $T \hookrightarrow \mathcal{T}$). Typically this will be one of the following cases

$$k^0 \hookrightarrow W^0 \qquad k^0 \hookrightarrow W^{\varnothing} \qquad k^0 \hookrightarrow V^{\sharp} \qquad k^0 \hookrightarrow \mathcal{S}$$

where we use the following notations for log schemes

$$\begin{aligned} k^0 &\quad - \quad (\operatorname{Spec} k, 1 \mapsto 0) \\ W^0 &\quad - \quad (\operatorname{Spec} W, 1 \mapsto 0) \\ W^{\varnothing} &\quad - \quad (\operatorname{Spec} W, \mathrm{triv}) \\ V^{\sharp} &\quad - \quad (\operatorname{Spec} V, \mathrm{can}) \\ \mathcal{S} &\quad - \quad (\operatorname{Spwf} W[\![s]\!], 1 \mapsto s) \end{aligned}$$

In the appropriate cases, we also have a lift of Frobenius.

Consider a fine log scheme (of Zariski type and of finite type) Y over T. We can lift it locally to a weak formal log scheme $\mathcal{Y}_\bullet$ (of Zariski type) over $\mathcal{T}$ and glue the lifting datum. By Yamada's work [**9**], we also have a good theory of coefficients in this context in the form of log overconvergent isocrystals $\mathrm{Isoc}^\dagger(Y/\mathcal{T})$ (respectively F-isocrystals $F\text{-}\mathrm{Isoc}^\dagger(Y/\mathcal{T})$) which can be interpreted as locally free sheaves with log connections on dagger spaces.

Now we define log rigid cohomology of Y over $\mathcal{T}$ with coefficients in $\mathscr{E} \in \mathrm{Isoc}^\dagger(Y/\mathcal{T})$ as a de Rham-type cohomology using the local lifting datum:

$$R\Gamma_{\mathrm{rig}}(Y/\mathcal{T}, \mathscr{E}) := R\Gamma(\mathfrak{Y}_\bullet, \mathscr{E}_{\mathfrak{Y}_\bullet} \otimes \omega^\star_{\mathcal{Y}_\bullet/\mathcal{T},\mathbb{Q}}).$$

1.1. Example. — We have the following local constructions: Consider the situation

$$\begin{aligned} Y &\quad - \quad \text{semistable over } k^0; \\ \mathcal{Z} &\quad - \quad \text{a lift to } \mathcal{S} \Rightarrow \text{log smooth over } W^{\varnothing}; \\ \mathcal{X} &\quad - \quad \mathcal{Z} \times V^{\sharp}; \\ \mathcal{Y} &\quad - \quad \mathcal{Z} \times W^0; \\ \mathfrak{Z}, \mathfrak{X}, \mathfrak{Y} &\quad - \quad \text{the associated dagger spaces.} \end{aligned}$$

We can compute different log rigid cohomologies

$$\begin{aligned} \omega^\bullet_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}} &\quad - \quad \text{computes the "absolute" rigid cohomolog } R\Gamma_{\mathrm{rig}}(Y/W^{\varnothing}); \\ \omega^\bullet_{\mathcal{Z}^{\varnothing}/W^{\varnothing},\mathbb{Q}} &\quad - \quad \text{computes the Berthelot's rigid cohomology } R\Gamma_{\mathrm{rig}}(Y^{\varnothing}/W^{\varnothing}) = R\Gamma_{\mathrm{rig}}(Y/F); \\ \omega^\bullet_{\mathcal{X}/V^{\sharp},\mathbb{Q}} &\quad - \quad \text{computes } R\Gamma_{\mathrm{rig}}(Y/V^{\sharp}); \\ \omega^\bullet_{\mathcal{Y}/W^0,\mathbb{Q}} &\quad - \quad \text{computes } R\Gamma_{\mathrm{rig}}(Y/W^0)\text{; should give Hyodo–Kato theory;} \\ \widetilde{\omega}^\bullet_{\mathcal{Y},\mathbb{Q}} &\quad - \quad \text{the auxiliary complex } \omega^\bullet_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}} \otimes_{\mathscr{O}_{\mathfrak{Z}}} \mathscr{O}_{\mathfrak{Y}}; \end{aligned}$$

Consider the so called Kim–Hain complexes (cf. [**7**]):

$$\omega^\bullet_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[u] \quad \text{and} \quad \widetilde{\omega}^\bullet_{\mathcal{Y},\mathbb{Q}}[u]$$

with $u^{[i]}$ of degree 0, such that $du^{[i+1]} = d\log s \cdot u^{[i]}$ and $u^{[0]} = 1$ and

— multiplication: $u^{[i]} \wedge u^{[j]} = \frac{(i+j)!}{i!j!} u^{[i+j]}$

— Frobenius: $\phi(u^{[i]}) = p^i u^{[i]}$

— monodromy: $N(u^{[i]}) = u^{[i-1]}$

The rigid Hyodo–Kato cohomology for Y/k semistable is given by $R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y) := R\Gamma(\mathfrak{Z}, \omega^{\bullet}_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[u])$ with endomorphisms φ and N, such that $N\varphi = p\varphi N$. This is justified by the following commutative diagram:

$$\begin{array}{ccccc}
R\Gamma(\mathfrak{Z}, \omega^{\bullet}_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[u]) & \longrightarrow & R\Gamma(\mathfrak{Z}, \omega^{\bullet}_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[\![u]\!]) & \xrightarrow[u^{[i]}\mapsto 0]{\sim} & R\Gamma(\mathfrak{Z}, \omega^{\bullet}_{\mathcal{Z}/\mathcal{S},\mathbb{Q}}) \\
\downarrow{\scriptstyle\sim} & & \downarrow & & \downarrow \\
R\Gamma(\mathfrak{Y}, \widetilde{\omega}^{\bullet}_{\mathcal{Y},\mathbb{Q}}[u]) & \xrightarrow{\sim} & R\Gamma(\mathfrak{Y}, \widetilde{\omega}^{\bullet}_{\mathcal{Y},\mathbb{Q}}[\![u]\!]) & \xrightarrow[u^{[i]}\mapsto 0]{\sim} & R\Gamma(\mathfrak{Y}, \omega^{\bullet}_{\mathcal{Y}/W^0,\mathbb{Q}})
\end{array}$$

Now define for a uniformiser $\pi \in V$ and $q \in \mathfrak{m}\backslash\{0\}$

$$\Psi_{\pi,q} : R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y) \to R\Gamma_{\mathrm{rig}}(Y/V^{\sharp})$$

induced by the natural morphism $\omega^{\bullet}_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}} \to \omega^{\bullet}_{\mathcal{Z}/\mathcal{S},\mathbb{Q}} \to \omega^{\bullet}_{\mathcal{X}/V^{\sharp},\mathbb{Q}}$ and $\Psi_{\pi,q}(u^{[i]}) := \frac{(-\log_q(\pi))^i}{i!}$. So the diagram now looks like:

$$\begin{array}{ccccc}
 & & R\Gamma_{\mathrm{rig}}(Y/W^{\varnothing}) & & \\
 & \swarrow \;\circlearrowleft & \downarrow & \circlearrowleft\; \searrow & \\
R\Gamma_{\mathrm{rig}}(Y/W^0) & \xleftarrow{\sim} & R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y) & \xrightarrow{\Psi_{\pi,q}} & R\Gamma_{\mathrm{rig}}(Y/V^{\sharp}, \pi), \\
 & \nwarrow \;\circlearrowleft & \downarrow & (*)\; \nearrow & \\
 & & R\Gamma_{\mathrm{rig}}(Y/\mathcal{S}) & &
\end{array}$$

where all triangles except for $(*)$ commute. The triangle $(*)$ commutes if $q = \pi$.

One of our main results is to show that $\Psi_{\pi,q}$ is independent of the choice of π, but depends on q, that for any π and q, it becomes a quasi-isomorphism after tensoring with K, and that it is compatible with Hyodo–Kato's original construction of the Hyodo–Kato map if $q = \pi$.

2. Monogenic sub log structures

As we will see, a good theory of coefficients is crucial for our definition of log rigid cohomology with compact support.

2.1. Definition. — For a (weak formal) scheme Y with a log structure $\mathcal{M}$ we say that $\mathcal{M}$ is *monogenic* if locally on Y there exists a chart of the form $\psi\colon \mathbb{N}_Y \to \mathcal{M}$. We call $\psi(1)$ a *(local) generator* of $\mathcal{M}$.

For a fine log scheme of Zariski type Y over the base log scheme T (where we denote the log structure by $\mathcal{N}_Y$, we will consider monogenic sub log structures $\mathcal{M} \subset \mathcal{N}_Y$. But as we want to consider log rigid cohomology we also have to consider a compatible sublog structure on a (local) lift to $\mathcal{T}$:

2.2. Construction. — *Let Y be a fine log scheme (of Zariski type and of finite type) over T with a lift $\mathcal{Y}$ to $\mathcal{T}$, that is, let $i : Y \hookrightarrow \mathcal{Y}$ be a homeomorphic exact closed immersion over $T \hookrightarrow \mathcal{T}$. For a monogenic sub log structure $\mathcal{M} \subset \mathcal{N}_Y$, we denote by $i_+\mathcal{M}$ the push forward as a sheaf of sets (not as a log structure), and let $i_\star\mathcal{M}$ be the preimage of $i_+\mathcal{M}$ under the morphism $\mathcal{N}_{\mathcal{Y}} \to i_+\mathcal{N}_Y$ induced by i. This is a monogenic sub log structure of $\mathcal{N}_{\mathcal{Y}}$.*

Now we can associate an overconvergent isocrystal to a monogenic substructure of a log structure.

2.3. Definition. — For a monogenic sub log structure $\mathcal{M} \subset \mathcal{N}_Y$ as above, let $\mathscr{O}_{\mathcal{Y}}(\mathcal{M})$ be the locally free $\mathscr{O}_{\mathcal{Y}}$-module locally generated by a generator of $i_\star\mathcal{M}$. The structure morphism $\alpha\colon \mathcal{N}_{\mathcal{Y}} \to \mathscr{O}_{\mathcal{Y}}$ induces a canonical $\mathscr{O}_{\mathcal{Y}}$-linear homomorphism

$$\mathscr{O}_{\mathcal{Y}}(\mathcal{M}) \to \mathscr{O}_{\mathcal{Y}}. \tag{2}$$

We denote by $\mathscr{O}_{\mathcal{Z}_\mathbb{Q}}(\mathcal{M})$ the locally free $\mathscr{O}_{\mathcal{Z}_\mathbb{Q}}$-module induced by $\mathscr{O}_{\mathcal{Z}}(\mathcal{M})$.

In general, Y only has a lift $\mathcal{Y}$ Zariski locally, but as the construction above glues, it induces after inverting p, an overconvergent isocrystal $\mathscr{O}^\dagger_{Y/\mathcal{T}}(\mathcal{M}) \in \mathrm{Isoc}^\dagger(Y/\mathcal{T})$ together with a canonical $\mathscr{O}^\dagger_{Y/\mathcal{T}}$-linear morphism

$$\mathscr{O}^\dagger_{Y/\mathcal{T}}(\mathcal{M}) \to \mathscr{O}^\dagger_{Y/\mathcal{T}}.$$

For any object $\mathscr{E} \in \mathrm{Isoc}^\dagger(Y/\mathcal{T})$, we define

$$\mathscr{E}(\mathcal{M}) := \mathscr{E} \otimes \mathscr{O}_{Y/\mathcal{T}}(\mathcal{M}).$$

2.4. Remark. — If the base $t \hookrightarrow \mathcal{T}$ has a lift of Frobenius σ, there is a canonical $\mathscr{O}^\dagger_{Y/\mathcal{T}}$-linear homomorphism

$$\sigma^* \mathscr{O}^\dagger_{Y/\mathcal{T}}(\mathcal{M}) \to \mathscr{O}^\dagger_{Y/\mathcal{T}}(\mathcal{M})$$

which is compatible with the canonical Frobenius structure $\mathscr{O}^\dagger_{Y/\mathcal{T}}(\mathcal{M}) \to \mathscr{O}^\dagger_{Y/\mathcal{T}}$ (but which is not in general a Frobenius structure, as it is not an isomorphism, if $\mathcal{M}$ is non-trivial).

Now we can define a compact support version of log rigid cohomology:

2.5. Definition. — Let $T \hookrightarrow \mathcal{T}$ and Y/T as anove and $\mathcal{M} \subset \mathcal{N}_Y$ a monogenic sub log structure. For any $\mathscr{E} \in \mathrm{Isoc}^\dagger(Y/\mathcal{T})$, we define the *log rigid cohomology with compact support towards* $\mathcal{M}$ of Y over $\mathcal{T}$ with coefficients in $\mathscr{E}$ to be

$$R\Gamma_{\mathrm{rig}, \mathcal{M}}(Y/\mathcal{T}, \mathscr{E}) := R\Gamma_{\mathrm{rig}}(Y/\mathcal{T}, \mathscr{E}(\mathcal{M})).$$

For the case $\mathscr{E} = \mathscr{O}_{Y/\mathcal{T}}$, we simply write

$$R\Gamma_{\mathrm{rig}, \mathcal{M}}(Y/\mathcal{T}) := R\Gamma_{\mathrm{rig}, \mathcal{M}}(Y/\mathcal{T}, \mathscr{O}_{Y/\mathcal{T}}).$$

Suppose that a Frobenius lift σ on $\mathcal{T}$ is given, then we obtain a σ-semilinear endomorphism

$$\varphi\colon R\Gamma_{\mathrm{rig}, \mathcal{M}}(Y/\mathcal{T}, \mathscr{E}) \to R\Gamma_{\mathrm{rig}, \mathcal{M}}(Y/\mathcal{T}, \mathscr{E}),$$

which is independent of local Frobenius liftings.

3. Compactly supported log rigid cohomology for semistable schemes

We now look specifically at the case of (strictly) semistable schemes over k^0, as in this case we have comparison morphisms to the crystalline version and moreover, we can establish Poincaré duality.

3.1. Definition. — (i) For integers $n \geqslant 1$ and $m \geqslant 0$, let $k^0(n,m) \to k^0$ be the morphism of fine log schemes induced by the commutative diagram

$$\begin{array}{ccc} \mathbb{N}^n \oplus \mathbb{N}^m & \xleftarrow{\beta} & \mathbb{N} \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha_0} \\ k[\mathbb{N}^n/\Delta(\mathbb{N}) \oplus \mathbb{N}^m] & \longleftarrow & k \end{array}$$

where $\Delta\colon \mathbb{N} \to \mathbb{N}^n$ is the diagonal map, α_0 is induced by the canonical chart of k^0, α is the composition of the natural morphisms $\mathbb{N}^n \oplus \mathbb{N}^m \to \mathbb{N}^n/\Delta(\mathbb{N}) \oplus \mathbb{N}^m \to k[\mathbb{N}^n/\Delta(\mathbb{N}) \oplus \mathbb{N}^m]$, and β is the composition of Δ and the canonical injection $\mathbb{N}^n \to \mathbb{N}^n \oplus \mathbb{N}^m$.

(ii) A log scheme Y over k^0 is called *strictly semistable*, if Zariski locally on Y there exists a strict log smooth morphism $Y \to k^0(n,m)$ over k^0. We let $\mathcal{M}_D \subset \mathcal{N}_Y$ be the monogenic log substructure locally generated by $(1,\ldots,1) \in \mathbb{N}^m$. The divisor D defined by local generators of $\mathcal{M}_D$ is called the *horizontal divisor* of Y. The horizontal divisor can be empty because we allow the case $m = 0$.

Specifically, we will consider the following situation: let $\overline{Y}$ be a proper strictly semistable log scheme over k^0 with horizontal divisor D. We want to define compactly supported log rigid cohomology theories for the complement $Y = \overline{Y} \backslash D$ which is a strictly semistable log scheme with empty horizontal divisor.

3.2. Definition. — Let $T \hookrightarrow \mathcal{T}$ be one of

$$k^0 \hookrightarrow W^0 \qquad k^0 \hookrightarrow W^{\varnothing} \qquad k^0 \hookrightarrow V^{\sharp} \qquad k^0 \hookrightarrow \mathcal{S}.$$

We define the *log rigid cohomology with compact support* of Y over $\mathcal{T}$ to be

$$R\Gamma_{\mathrm{rig},c}(Y/\mathcal{T}) := \quad R\Gamma_{\mathrm{rig},\mathcal{M}_D}(\overline{Y}/\mathcal{T}).$$

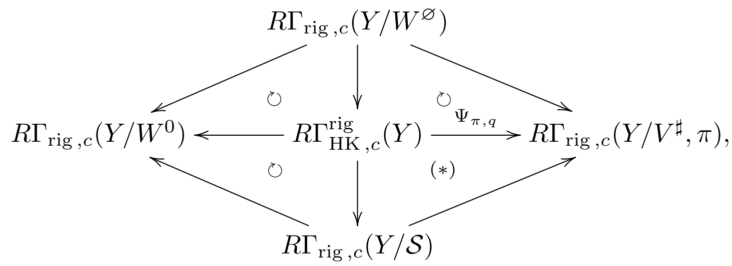
If the base is in particular $k^0 \hookrightarrow W^{\varnothing}$, we define the *rigid Hyodo–Kato cohomology with compact support* of Y to be

$$R\Gamma^{\mathrm{rig}}_{\mathrm{HK},c}(Y) \quad := \quad R\Gamma^{\mathrm{rig}}_{\mathrm{HK},\mathcal{M}_D}(\overline{Y})$$

As in the case without compact support, we can define the Hyodo–Kato morphism

$$\Psi_{\pi,q} : R\Gamma^{\mathrm{rig}}_{\mathrm{HK},c}(Y) \to R\Gamma_{\mathrm{rig},c}(Y/V^{\sharp})$$

for a uniformiser $\pi \in V$ and $q \in \mathfrak{m}\backslash\{0\}$, and obtain again a diagram



where all triangles except for $(*)$ commute and the triangle $(*)$ commutes if $q = \pi$.

Note that the definition of log rigid cohomology with compact support depends on a good theory of coefficients for log rigid cohomology. However, we haven't established the necessary quasi-isomorphisms (for example of $\Psi_{\pi,q} \otimes K$), for general coefficients. Thus, to establish these properties for the compactly supported cohomology, we took advantage of the special nature of the coefficients used which allowed us to reduce to the non-compactly supported case.

3.3. Construction. — Let $Y \subset \overline{Y} \supset D$ be as above, and consider the decompositions into irreducible components

$$\overline{Y} = \bigcup_{i \in \Upsilon_{\overline{Y}}} \overline{Y}_i \qquad\qquad D = \bigcup_{j \in \Upsilon_D} D_j.$$

For $J \subset \Upsilon_D$, let $D_J := \bigcap_{j \in J} D_j$. Denote by $\overline{Y}^{\flat}$ the log scheme whose underlying scheme is that of $\overline{Y}$ and whose log structure is generated by log substructures corresponding to all $i \in \Upsilon_{\overline{Y}}$, and endow D_J with the pull-back log structure from $\overline{Y}^{\flat}$, denoted by $D_J^{\flat}$. We obtain a simplicial log scheme $D^{(\bullet),\flat}$ by setting for each $r \in \mathbb{N}$

$$D^{(r),\flat} := \coprod_{J \subset \Upsilon_D, |J| = r} D_J^{\flat}.$$

Now we could obtain the following canonical quasi-isomoprhisms

$$\begin{aligned} &R\Gamma_{\mathrm{rig},c}(Y/W^{\varnothing}) \xrightarrow{\sim} R\Gamma_{\mathrm{rig}}(D^{(\bullet),\flat}/W^{\varnothing}),\\ &R\Gamma^{\mathrm{rig}}_{\mathrm{HK},c}(Y) \xrightarrow{\sim} R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(D^{(\bullet),\flat}),\\ &R\Gamma_{\mathrm{rig},c}(Y/\mathcal{S}) \xrightarrow{\sim} R\Gamma_{\mathrm{rig}}(D^{(\bullet),\flat}/\mathcal{S}),\\ &R\Gamma_{\mathrm{rig},c}(Y/W^0) \xrightarrow{\sim} R\Gamma_{\mathrm{rig}}(D^{(\bullet),\flat}/W^0),\\ &R\Gamma_{\mathrm{rig},c}(Y/V^{\sharp})_{\pi} \xrightarrow{\sim} R\Gamma_{\mathrm{rig}}(D^{(\star),\flat}/V^{\sharp})_{\pi}.\end{aligned}$$

We use this to transfer the properties that we have already shown for the non-compcatly supported log rigid cohomology theories to the compactly supported version:

3.4. Theorem. — *Let Y be strictly semistable over k^0 (with empty horizontal divisor) and assume it has a compactification $\overline{Y}$ which is strictly semistable over k^0 with horizontal divisor D.*

(i) The map $R\Gamma^{\mathrm{rig}}_{\mathrm{HK},c}(Y) \to R\Gamma_{\mathrm{rig},c}(Y/W^0)$ is a quasi-isomorphism.

(ii) We have quasi-isomorphisms

$$\begin{aligned} &R\Gamma_{\mathrm{rig},c}(Y/\mathcal{S}) \otimes_{F\{s\},s\mapsto 0} F \xrightarrow{\cong} R\Gamma_{\mathrm{rig},c}(Y/W^0),\\ &R\Gamma_{\mathrm{rig},c}(Y/\mathcal{S}) \otimes_{F\{s\},s\mapsto \pi} K \xrightarrow{\cong} R\Gamma_{\mathrm{rig},c}(Y/V^{\sharp}),\\ &R\Gamma^{\mathrm{rig}}_{\mathrm{HK},c}(Y) \otimes_F F\{s\} \xrightarrow{\cong} R\Gamma_{\mathrm{rig},c}(Y/\mathcal{S}).\end{aligned}$$

(iii) For any $q \in \mathfrak{m}\backslash\{0\}$, the map $\Psi_{\pi,q}$ induces a quasi-isomorphism $\Psi_{\pi,q,K}\colon R\Gamma^{\mathrm{rig}}_{\mathrm{HK},c}(Y) \otimes_F K \xrightarrow{\cong} R\Gamma_{\mathrm{rig},c}(Y/V^{\sharp})$.

(iv) The cohomology groups $H^*_{\mathrm{rig},c}(Y/W^{\varnothing})$, $H^{\mathrm{rig},*}_{\mathrm{HK},c}(Y)$, $H^i_{\mathrm{rig},c}(Y/W^0)$ are finite dimensional F-vector spaces, $H^*_{\mathrm{rig},c}(Y/\mathcal{S})$ is a free $F\{s\}$-module of finite rank, and $H^*_{\mathrm{rig},c}(Y/V^{\sharp})$ is a finite dimensional K-vector space. The monodromy operator on $H^{\mathrm{rig},*}_{\mathrm{HK},c}(Y)$ is nilpotent.

(v) The induced Frobenius operator φ on $H^{\mathrm{rig},*}_{\mathrm{HK},c}(Y,)$ is bijective.

Our construction is compatible with Berthelot's compactly supported rigid cohomology in the smooth case:

3.5. Proposition. — *Let Y be smooth over k and assume that it has a simple normal crossing compactification $\overline{Y}$ (i.e. $\overline{Y}$ is a projective smooth scheme over k and $D = \overline{Y}\backslash Y$ a simple normal crossing divisor). Then*

$$R\Gamma_{\mathrm{rig},c}(Y^{\varnothing}/W^{\varnothing}) \xrightarrow{\sim} R\Gamma^{\mathrm{Ber}}_{\mathrm{rig},c}(Y/F).$$

where the left hand side is computed by $R\Gamma_{\mathrm{rig},\mathcal{M}_D}(\overline{Y}^{\infty}/W^{\varnothing})$ where $\overline{Y}^{\infty}$ is the log scheme whose underlying scheme is $\overline{Y}$ with log structure induced by D. Hence the pull-back log structure on Y is the trivial log structure.

We also established comparison morphism with the log crystalline cohomology theory over W^0 *compactly supported towards a simple normal crossing divisor* of Tsuji [**8**] (see also [**10**]) in certain cases. This is rather important to establish Poincaré duality.

4. Poincaré duality

Let Y be strictly semistable over k^0 (with empty horizontal divisor) and assume it has a compactification $\overline{Y}$ which is strictly semistable over k^0 with horizontal divisor D. We have pairings for the different rigid cohomology theories (compatible with Frobenius and monodromy in the appropriate cases)

$$R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y)) \otimes R\Gamma^{\mathrm{rig}}_{\mathrm{HK},c}(Y) \to R\Gamma^{\mathrm{rig}}_{\mathrm{HK},c}(Y),$$
$$R\Gamma_{\mathrm{rig}}(Y/W^0) \otimes R\Gamma_{\mathrm{rig},c}(Y/W^0) \to R\Gamma_{\mathrm{rig},c}(Y/W^0),$$
$$R\Gamma_{\mathrm{rig}}(Y/W^{\varnothing}) \otimes R\Gamma_{\mathrm{rig},c}(Y/W^{\varnothing}) \to R\Gamma_{\mathrm{rig},c}(Y/W^{\varnothing}),$$
$$R\Gamma_{\mathrm{rig}}(Y/V^{\sharp}) \otimes R\Gamma_{\mathrm{rig},c}(Y/V^{\sharp}) \to R\Gamma_{\mathrm{rig},c}(Y/V^{\sharp}).$$

Because of the comparison isomorphisms to Tsuji's crystalline version mentioned above, Poincaré duality for log crystalline cohomology over W^0 [**8**] carries over to log rigid cohomology over W^0. From there one can then deduce Poincaré duality for all the other log rigid cohomology theories due to the compatibilities from the theorem in the previous section.

To summarise, we obtain the following theorem.

4.1. Theorem. — *Let $d := \dim Y$. Then there exist canonical isomorphisms*

$$R\Gamma_{\mathrm{rig}}(Y/W^{\varnothing}) \xrightarrow{\sim} R\Gamma_{\mathrm{rig},c}(Y/W^{\varnothing})^*[-2d-1](-d-1) \qquad \text{in } D^b(\mathrm{Mod}^{\mathrm{fin}}_F(\varphi)),$$
$$R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y) \xrightarrow{\sim} R\Gamma^{\mathrm{rig}}_{\mathrm{HK},c}(Y)^*[-2d](-d) \qquad \text{in } D^b(\mathrm{Mod}^{\mathrm{fin}}_F(\varphi, N)),$$
$$R\Gamma_{\mathrm{rig}}(Y/W^0) \xrightarrow{\sim} R\Gamma_{\mathrm{rig},c}(Y/W^0)^*[-2d](-d) \qquad \text{in } D^b(\mathrm{Mod}^{\mathrm{fin}}_F(\varphi)),$$
$$R\Gamma_{\mathrm{rig}}(Y/V^{\sharp}) \xrightarrow{\sim} R\Gamma_{\mathrm{rig},c}(Y/V^{\sharp})^*[-2d] \qquad \text{in } D^b(\mathrm{Mod}^{\mathrm{fin}}_K),$$

where $*$ denotes the (derived) internal Hom $R\underline{\mathrm{Hom}}(-,F)$ (or $R\underline{\mathrm{Hom}}(-,K)$). The Hyodo–Kato maps are compatible with Poincaré duality in the sense that $\Psi^*_{\pi,q,K} = \Psi^{-1}_{\pi,q,K}$.

4.2. Remark. — Poincaré duality as stated above implies that the compactly supported log rigid cohomology defined in this paper is independent of the compactification (as long as it is a simple normal crossing compactification).

End

Thank you very much for your attention!

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