A Rigid Analytic Approach to Hyodo–Kato Theory

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Outline

1. Periods and Comparison Theorems
   - p-adic Numbers
   - Étale cohomology and Galois Representations
   - p-adic Comparison Theorems
2. Hyodo–Kato theory
   - Classical constructions
   - Rigid analytic construction

A Classical Comparison Theorem

$M/\mathbb{C}$: complex manifold

Theorem (Complex de Rham Theorem)

There is a non-degenerate pairing

$$H^i_{\mathrm{dR}}(M) \times H_i(M, \mathbb{C}) \to \mathbb{C}, (\omega, \gamma) \mapsto \int_\gamma \omega.$$

de Rham cohomology : $H^i_{\mathrm{dR}}(M) := H^i(M, \Omega^\bullet_M)$

singular homology : $H_i(M, \mathbb{C})$

Dually:

$$H^i_{\mathrm{dR}}(M) \cong H^i(M, \mathbb{C}).$$

If M is a compact Kähler manifold: Hodge decomposition $H^i(M, \mathbb{C}) = \bigoplus_{p+q=i} H^q(M, \Omega^p_M)$.

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$Z/\mathbb{Q}$: smooth, projective algebraic variety, (gives rise to a complex manifold)

Theorem (Algebro-geometric de Rham Theorem)

There is a non-degenerate pairing

$$H^i_{\mathrm{dR}}(Z_{\mathbb{C}}) \times H_i(Z(\mathbb{C}), \mathbb{C}) \to \mathbb{C}, (\omega, \gamma) \mapsto \int_\gamma \omega.$$

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Dually:

$$H^i_{\mathrm{dR}}(Z) \otimes_{\mathbb{Q}} \mathbb{C} \cong H^i(Z(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$$

$\mathbb{C}$ contains periods for all varieties! Example: $\int_\gamma \frac{\mathrm{d}z}{z} = 2\pi i$.

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Completions of the Rational Numbers

Obtain $\mathbb{C}$ from $\mathbb{Q}$ by completion wrt the **archimedean norm** $|\cdot|$ on $\mathbb{Q}$:

$$\mathbb{Q} \hookrightarrow \widehat{\mathbb{Q}} \cong \mathbb{R} \hookrightarrow \mathbb{C} \cong \overline{\mathbb{R}}$$

Archimedean completion.

But also **non-archimedean norms**!

p a prime number $\Rightarrow$ the p-adic norm $|\cdot|_p$: for $x \in \mathbb{Q}$: $|x|_p = \left(\frac{1}{p}\right)^{\operatorname{ord}_p(x)}$

E.g.: $x = \frac{28}{3} = 2^2 \cdot 7^1 \cdot 3^{-1} \Rightarrow p = 2$: $\left|\frac{28}{3}\right|_2 = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$

$\Rightarrow p = 3$: $\left|\frac{28}{3}\right|_3 = \left(\frac{1}{3}\right)^{-1} = 3$,

$\Rightarrow p = 5$: $\left|\frac{28}{3}\right|_5 = \left(\frac{1}{5}\right)^0 = 1$

Satisfies $|xy| = |x||y|$ and $|x + y| \leqslant \max(|x|, |y|)$.

$$\mathbb{Q} \mapsto \widehat{\mathbb{Q}} \cong \mathbb{Q}_p \hookrightarrow \overline{\mathbb{Q}}_p \hookrightarrow \widehat{\overline{\mathbb{Q}}}_p = \mathbb{C}_p$$

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$\mathbb{Q}_p$ – completion of $\mathbb{Q}$ via $|\cdot|_p$,

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p \,\big|\, |x|_p \leq 1\}, \quad \mathbb{Z}_p \cong \varprojlim \mathbb{Z}/p^n,$$
$$\mathbb{Z}_p\text{"} = \text{"}\{0, 1, \ldots, p-1\}[\![p]\!],$$
$$\mathbb{Q}_p = \mathbb{Z}_p[1/p], \quad \mathbb{Q}_p \ni x = \sum_{n \geqslant n_0} x_n p^n, x_n \in \{0, \ldots, p-1\}.$$

$\overline{\mathbb{Q}}_p$ – algebraic closure of $\mathbb{Q}_p$,

$|\cdot|_p$ extends uniquely to $\overline{\mathbb{Q}}_p$,

$G_{\mathbb{Q}_p} := \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts via isometries

$\overline{\mathbb{Q}}_p$ is not complete for $|\cdot|_p$,

$\mathbb{C}_p$ – the completion of $\overline{\mathbb{Q}}_p$ via $|\cdot|_p$,

$G_{\mathbb{Q}_p} = \mathrm{Aut}_{cont}(\mathbb{C}_p)$,

$\dim_{\mathbb{Q}_p} \mathbb{C}_p$ is not countable, $\mathbb{C}_p \cong \mathbb{C}$ as an abstract field.

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Étale cohomology

Question

*Is there a p-**adic analogue** of de Rham's theorem?*

For p-adic coefficients, we have

$$H^i(Z(\mathbb{C}), \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong H^i_{\text{et}}(Z_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$$

$H^i_{\text{et}}(Z_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ – Grothendieck's étale cohomology,
finite rank over $\mathbb{Q}_p$,
continuous action of $G_{\mathbb{Q}_p} := \text{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

This action carries information about:

1. finite extensions of $\mathbb{Q}_p$,
2. the arithmetic of Z, for example its rational points $Z(\mathbb{Q})$.

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Example: Cyclotomic Character

ζ_n : a primitive p^nth root of unity,

- p^nth roots of unity correspond to elements in $\mathbb{Z}/p^n$
- primitive p^nth roots of unity correspond to elements in $(\mathbb{Z}/p^n)^*$
- every p^nth root of unity is a power of ζ_n
- an element $g \in G_{\mathbb{Q}_p}$ sends ζ_n to another primitive p^nth root of unity: $g(\zeta_n) = \zeta_n^{a_{g,n}}$ with $a_{g,n} \in (\mathbb{Z}/p^n)^*$

The cyclotomic character is defined as

$$\chi : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^* = \varprojlim (\mathbb{Z}/p^n)^*, \quad g \mapsto (a_{g,n})_n.$$

$\mathbb{Z}_p$-module $\mathbb{Z}_p(1) = \mathbb{Z}_p \cdot \zeta$ with Galois action:

$$\lambda \cdot \zeta = (\zeta_n^{\lambda_n})_n, \qquad \lambda \in \mathbb{Z}, \quad \lambda_n \equiv \lambda \mod p^n\mathbb{Z}_p$$

$$g(\zeta) = \chi(g) \cdot \zeta = (\zeta_n^{a_{g,n}})_n, \qquad g \in G_{\mathbb{Q}_p}$$

If $r \in \mathbb{Z}$, $\mathbb{Q}_p(r)$ is $\mathbb{Q}_p$ with action of $G_{\mathbb{Q}_p}$ via χ^r.

Realisation via étale cohomology of $\mathbb{P}^1$: $\quad \mathbb{Q}_p(1) \cong H^2_{\text{et}}(\mathbb{P}^1_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)^*$.

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p-adic Period Rings

Question

Is there a p-adic period ring B containing periods of all varieties over $\mathbb{Q}_p$ such that

1. *there is an isomorphism*

$$H^i_{\mathrm{dR}}(Z) \otimes_{\mathbb{Q}_p} B \cong H^i_{\mathrm{et}}(Z_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B,$$

2. *we can recover the Galois representation $H^i_{\mathrm{et}}(Z_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p)$ from $H^i_{\mathrm{dR}}(Z)$?*

B cannot be $\mathbb{C}_p$: $\mathbb{C}_p$ does not contain a p-adic analog of $2\pi i$ (Tate, '66).

Fontaine ('80) constructed a filtered ring $\mathbf{B}^+_{\mathrm{dR}}$, with Galois action

$$2\pi i = t \in \mathbf{B}^+_{\mathrm{dR}}, \quad F^n\mathbf{B}^+_{\mathrm{dR}} := (t^n), \quad \mathrm{gr}^n_F \mathbf{B}^+_{\mathrm{dR}} = \mathbb{C}_p(n)$$

Define $\mathbf{B}_{\mathrm{dR}} := \mathbf{B}^+_{\mathrm{dR}}[1/t]$.

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Fontaine ('80) constructed a filtered ring $\mathbf{B}^+_{\mathrm{dR}}$, with Galois action

$$2\pi i = t \in \mathbf{B}^+_{\mathrm{dR}}, \quad F^n\mathbf{B}^+_{\mathrm{dR}} := (t^n), \quad \mathrm{gr}^n_F \mathbf{B}^+_{\mathrm{dR}} = \mathbb{C}_p(n)$$

Define $\mathbf{B}_{\mathrm{dR}} := \mathbf{B}^+_{\mathrm{dR}}[1/t]$.

p-adic Period Rings

Question

Is there a p-adic period ring B containing periods of all varieties over $\mathbb{Q}_p$ such that

1. *there is an isomorphism*

$$H^i_{\mathrm{dR}}(Z) \otimes_{\mathbb{Q}_p} B \cong H^i_{\mathrm{et}}(Z_{\overline{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B,$$

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de Rham Comparison

Theorem (Faltings '89)

Z – proper, smooth over K, $[K:\mathbb{Q}_p] < \infty$. There is an isomorphism

$$\alpha_{\mathrm{dR}}: \quad H^i_{\mathrm{dR}}(Z) \otimes_K \mathbf{B}_{\mathrm{dR}} \cong H^i_{\mathrm{et}}(Z_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}}$$

compatible with Galois action and filtration.

Take $\mathrm{gr}^0_F \Rightarrow$ a Hodge–Tate decomposition:

$$H^i(Z_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_p \cong \bigoplus_{r \geq 0} H^{i-r}(Z, \Omega^j_{Z/K}) \otimes_K \mathbb{C}_p(-r)$$

Take G_K-fixed points $\Rightarrow$ recover H^i_{dR}:

$$H^i_{\mathrm{dR}}(Z) \cong (H^i_{\mathrm{et}}(Z_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}})^{G_K}, \quad + \text{Fil.}$$

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$\Rightarrow$ Crystalline Conjecture (Fontaine),
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History: Fontaine-Messing, Hyodo, Kato, Faltings, Tsuji, Nizioł ('85-2005); Beilinson, Bhatt, Scholze (2010+).

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p-adic Period Isomorphism

Theorem

Z/K variety. There is an isomorphism

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compatible with Frobenius, monodromy, Galois action and with the de Rham period isomorphism α_{dR}.

Now we can go the other way:

$$H^i_{\mathrm{et}}(Z_{\overline{K}}, \mathbb{Q}_p) \cong (H^i_{\mathrm{HK}}(Z_{\overline{K}}) \otimes_{K^{nr}} \mathbf{B}_{\mathrm{st}})^{N=0,\phi=1} \cap F^0(H^i_{\mathrm{dR}}(Z_{\overline{K}}) \otimes_{\overline{K}} \mathbf{B}_{\mathrm{dR}})$$

Hyodo–Kato cohomology is a key object in the formulation of semistable conjecture.

$\Rightarrow$ plays an important role in several areas of arithmetic geometry, e.g. the research of **special values of L-functions**.

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Frobenius and Monodromy on de Rham cohomology

K – finite extension of $\mathbb{Q}_p$
V – ring of integers of K
$\mathfrak{m}$ – its maximal ideal
k – its residue field (perfect of characteristic $p > 0$)
$W(k)$ - ring of Witt vectors
F - its fraction field

Assume that Z has a "nice" integral model X/V, e.g. smooth or semistable. Denote X_0/k its special fibre, $X_K = Z$ its generic fibre.

What we want:

*Endow the de Rham cohomology $H^i_{\mathrm{dR}}(X_K)$ with a Frobenius (and monodromy) to obtain a **filtered φ-module** or **filtered (φ, N)-module** via comparison to a "richer" cohomology theory.*

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Hyodo–Kato like Cohomologies

X smooth, proper	crystalline cohomology $H^i_{\mathrm{cris}}(X_0/W(k))$	Grothendieck, Berthelot
X smooth	rigid cohomology $H^i_{\mathrm{rig}}(X_0/F)$	Berthelot
X semistable, proper	log-crystalline cohomology $H^i_{\log\,\mathrm{cris}}(X_0/W(k))$	Hyodo, Kato
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1. The crystalline versions provide **integral** theories - finite $W(k)$-modules. The rigid versions are only rational, but more versatile.
2. The rigid versions are more **computable** – use rigid analytic methods.
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It is highly non trivial to obtain a functorial homomorphism

$$\Psi : H^i_{\mathrm{HK}}(X) \to H^i_{\mathrm{dR}}(X_K)$$

which is an isomorphism after $\otimes K$.

1. Hyodo–Kato – the original (crystalline) definition Ψ^{HK}_π, depends on the choice of a uniformiser.
2. Beilinson – new representation of Hyodo–Kato complex, Hyodo–Kato morphism Ψ^B independent of the choice of a uniformiser – abstract crystalline construction.
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Our construction (joint with Kazuki Yamada)

Let X/V be semistable.

1. We construct a new representation of Hyodo–Kato cohomology with monodromy and Frobenius

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- The construction uses (a refined version of) weak formal schemes and dagger spaces.⇒ It is computable!
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Theorem (E–Yamada)

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Outlook

1. A version with compact supports?
   ⇒ Log rigd syntomic cohomology with compact supports.
2. Extension to K-varieties?
   ⇒ Like Nekovář–Nizioł's construction, but more computable.
3. Applications: special values of L-functions, comparison of rigid and log rigid cohomology via the monodromy,...

Dank u wel!

Thank you very much for your attention!