The infinity category of spectra

Preliminaries to understand the work of Beilinson on
Relative continuous K-theory and cyclic homology
Kleinwalsertal-Workshop

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Abstract

These are the notes for my talk at the workshop on "Continuous K-theory of p-adic rings" in Kleinwalsertal, September 2014, organised by Prof. Dr. Moritz Kerz. The goal of this workshop was to understand recent work of Prof. Alexander Beilinson on the subject. In this talk I presented some basic preliminaries about triangulated categories and spectra. It was continued by Oriol Raventos, who explained the pro-version of the concepts mentioned here.

Résumé

Voilà öes notes pour mon exposé pour le séminaire "Continuous K-theory of p-adic rings" dans Kleinwalsertal, en septembre 2014, organisé par Prof. Dr. Moritz Kerz. Le but de ce séminaire était de comprendre du travail récent de Prof. Alexander Beilinson sur ce sujet. Dans cet exposé je présente élément préliminaire sur les catégories trianguléées et les spectres. La série a été continuée par Oriol Raventos qui a décrit la version de pro-objets.

Key Words : Triangulated categories, Verdier quotient, Spectra.
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Contents

1 Triangulated categories

1.1 Definitions

We give a brief recollection of triangulated categories and Verdier quotients following [4]. The concept of triangulated categories was formally first introduced by Verdier. Essentially this is an additive category $\mathscr{T}$ with an additiv automorphism

$$\Sigma : \mathscr{T} \to \mathscr{T}$$

and a class of "distinguished" triangles (that play in a sense the role of short exact sequences.

Definition 1.1.1. A triangle in $\mathscr{T}$ is a diagram

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X.$$

We call it a candidate triangle if $v \circ u$, $w \circ v$ and $\Sigma u \circ w$ are all trivial. A morphism of triangle is a map that make the obvious diagrams commute.

A class of triangles in $\mathscr{T}$ is distinguished if

TR0 Trianges isomorphic to distinguished triangles are distinguished ($\Rightarrow$ up to non-unique isomorphism!) The trivial triangle

$$X \xrightarrow{\text{id}} X \to 0 \to \Sigma X$$

is distinguished.

TR1 If $f : X \to Y$ is a morphism in $\mathscr{T}$, then there exists a triangle

$$X \xrightarrow{f} Y \to Z \to \Sigma X$$

where Z is called the mapping cone of f.

TR2 Consider

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} \Sigma X$$

and the "reverse" triangle

$$Y \xrightarrow{-v} Z \xrightarrow{-w} \Sigma X \xrightarrow{-\Sigma u} \Sigma Y$$

One is distinguished if and only if the other is distinguished.

TR3 Consider a commutative diagram with morphisms f and g

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & & & \downarrow{\scriptstyle \Sigma f} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

then there is a morphism $h : Z \to Z'$ that makes the diagram commute.

Remark 1.1.2. With these properties one can show that distinguished triangles are candidate triangles.

This provides a pretriangulated category. If in addition

TR4' For any diagram of distinguished triangles

$$\begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma X \\ {\scriptstyle f}\downarrow & & {\scriptstyle g}\downarrow & & {\scriptstyle \exists h}\downarrow & & \downarrow{\scriptstyle \Sigma f} \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma X' \end{array}$$

h can be chosen in a way such that the mapping cone

$$Y \oplus X' \to Z \oplus Y' \to \Sigma X \oplus Z' \to \Sigma Y \oplus \Sigma X'$$

is distinguished.

holds, then $\mathscr{T}$ is called triangulated.

Remark 1.1.3. Some authors claim that this might not be the "right" definition, as it is only defined up to non-unique isomorphism.

Remark 1.1.4. The property **TR4'** can be shown to be equivalent to the more classical octahedral axiom **TR4**.

We discuss some properties of triangulated categories.

- The additive functor Σ preserves products and coproducts (because it is invertible, giving a right/left adjoint).
- A functor $H : \mathscr{T} \to \mathscr{A}$ into some abelian category is called homological (or cohomological for $\mathscr{T}^{\mathrm{op}}$) if it takes distinguished triangles to exact sequences. Examples are $\mathrm{Hom}(U, -)$ and $\mathrm{Hom}(-, U)$ for $U \in \mathscr{T}$.
- A square

$$\begin{array}{ccc} Y & \xrightarrow{f} & Z \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g'} \\ Y' & \xrightarrow[f']{} & Z' \end{array}$$

is called homotopy cartesian, if there is a distinguished triangle

$$Y \to Y' \oplus Z \to Z' \xrightarrow{\partial} \Sigma Y$$

for some map $\partial : Z' \to \Sigma Y$. In this case Y is the homotopy pullback and Z' is the homotopy pushout.

- A triangulated subcategory is a subcategory that is closed under isomorphism, closed under Σ and closed under extension (for a distinguished triangle $X \to Y \to Z \to \Sigma X$ in $\mathscr{T}$, if $X, y \in \mathscr{S} \subset \mathscr{T}$ then $Z \in \mathscr{S}$).

We can consider the following additional conditions.

TR5 Arbitrary coproducts exist in $\mathscr{T}$.

TR5$*$ Arbitrary products exist in $\mathscr{T}$.

Assume **TR5** (in fact the existence of countable coproducts would be sufficient), let

$$X_0 \xrightarrow{j_1} X_1 \xrightarrow{j_2} X_3 \to \cdots$$

be a sequence in $\mathscr{T}$. Then the homotopy colimit of this sequence exists and is defined by a triangle

$$\coprod X_i \xrightarrow{1-\mathrm{shift}} \coprod \to \mathrm{hocolim}\, X_i \to \Sigma \left(\coprod X_i \right),$$

where "shift" is given by the sum of the j_i's.
Similar for the homotopy limit with **TR5**$*$.

This notion is not exactly functorial but useful.

1.2 Verdier quotients

Let $\mathscr{D}_1$ and $\mathscr{D}_2$ be triangulated categories. A functor $F : \mathscr{D}_1 \to \mathscr{D}_2$ between them is called exact or triangulated, if it commutes with Σ up to isomorphism, i.e.

$$\forall X \exists \text{ natural isomorphism } \Phi_X : F(\Sigma X) \to \Sigma(F(X)),$$

and it takes distinguished triangles to distinguished triangles.

Let $\mathscr{D}$ be triangulated and $\mathscr{C} \subset \mathscr{D}$ a full additive subcategory. It is called triangulated subcategory, if

- it is closed under isomorphism
- the inclusion functor is an exact functor in the sense above
- the inclusion commutes with Σ ($\Phi_X : 1(\Sigma X) \to \Sigma(1(X)) = \mathrm{id}_{\Sigma X}$)

If $F : \mathscr{D} \to \mathscr{T}$ is triangulated, one can define the kernel of F

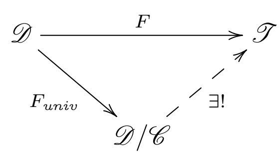
$$\ker F = \left\{ x \in \mathscr{D} \quad \middle| \quad F(X) \cong 0 \right\}$$

Some facts:

- $\ker F$ is a full subcategory of $\mathscr{D}$
- $\ker F$ is triangulated
- $\ker F$ contains all direct summands of its objects

If a subcategory satisfies these conditions, it is called a thick subcategory.

Theorem 1.2.1 (Verdier)**.** *Let $\mathscr{C} \subset \mathscr{D}$ be a triangulated subcategory. There exists a triangulated category denoted by $\mathscr{D}/\mathscr{C}$ and an exact functor $F_{univ} : \mathscr{D} \to \mathscr{D}/\mathscr{C}$ such that it is universal for $\mathscr{C} \subset \ker F_{univ}$. More precisely, if there is another exact functor $F : \mathscr{D} \to \mathscr{T}$ of triangulated categories such that $\mathscr{C} \subset \ker F$ then there is a unique functor $\mathscr{D}/\mathscr{C} \to \mathscr{T}$ that makes the diagram*



commute.

Remark 1.2.2. The category $\ker F_{\mathrm{univ}}$ is the smalls thick subcategory of $\mathscr{D}$ containing $\mathscr{C}$. If $\mathscr{C}$ is thick, then $\mathscr{C} = \ker F_{\mathrm{univ}}$.

This is called the verdure quotient. The functor F_{univ} is universal for functors that invert all morphisms of triangles in $\mathscr{C}$. The category $\mathscr{D}/\mathscr{C}$ is additive, and the functor F_{univ} is an additive functor.

2 Isogenies

We give a brief overview of isogenies as presented in [1, Section 1.1]. We will see, that it differs slightly from the definition one might be used to.

Let $\mathscr{C}$ be an additive category. We say that $n \in \mathbb{N}$ kills $X \in \mathscr{C}$ if $n \, \mathrm{id}_X = 0$. In this case, X is called a bounded torsion object.

Definition 2.0.3. A map $f : X \to Y$ in $\mathscr{C}$ is an isogeny, if there exists a map $g : Y \to X$ in $\mathscr{C}$ such that

$$f \circ g = n \, \mathrm{id}_Y \qquad \text{and} \qquad g \circ f = n \, \mathrm{id}_X \qquad \text{for some} \quad n \neq 0.$$

Now we can localise $\mathscr{C}$ with respect to isogenies and get a new category denoted $\mathscr{C} \otimes \mathbb{Q}$. It comes together with a functor

$$\mathscr{C} \to \mathscr{C}_{\mathbb{Q}}$$

which is bijective on objects (i.e. $X_{\mathbb{Q}}$ is the object up to isogeny corresponding to X), and on morphisms, we identify

$$\mathrm{Hom}(X_{\mathbb{Q}}, Y_{\mathbb{Q}}) = \mathrm{Hom}(X, Y) \otimes \mathbb{Q} \,.$$

Thus f is an isogeny, if and only if $f_{\mathbb{Q}}$ is invertible.

Example 2.0.4.

$$\begin{aligned} X_{\mathbb{Q}} = 0 &\Leftrightarrow X \text{ isogenous to } 0 \\ &\Leftrightarrow \exists X \to 0 \quad \wedge \quad 0 \to X \quad : \quad gf = n\,\mathrm{id}_0 \text{ and } fg = n\,\mathrm{id}_X \\ &\Leftrightarrow \exists n \neq 0 \quad : \quad n\,\mathrm{id}_X = 0 \end{aligned}$$

Let $\mathscr{I}$ be another category, and $G, F : \mathscr{I} \to \mathscr{C}$ functors. An isogeny of functors can be defined in essentially the same way as for categories.

Definition 2.0.5. A morphism of functors $f : F \to G$ is an isogeny, if there is a morphism of functors $g : G \to F$ such that

$$f \circ g = n\,\mathrm{id}_G \qquad \text{and} \qquad g \circ f = n\,\mathrm{id}\,F \quad \text{for some } n \neq 0.$$

Remark 2.0.6. If $\mathscr{I}$ is essentially small, then f is an isogeny in the category $\mathscr{C}^{\mathscr{I}}$ in the sense above.

If $\mathscr{C}$ is abelian, then the newly defined category $\mathscr{C} \otimes \mathbb{Q}$ is also abelian, and in this case, it coincides with $\mathscr{C}_{\mathbb{Q}}$, which is the category $\mathscr{C}$ modulo the Serre subcategory of bounded torsion objects. Moreover, if $\mathscr{C}$ is a tensor category, $\mathscr{C}_{\mathbb{Q}}$ is one as well. This is, because bounded torsion objects form an ideal. Thus $\mathscr{C} \to \mathscr{C}_{\mathbb{Q}}$ is a tensor functor.

Example 2.0.7. Consider the category $\mathcal{Ab}$ of abelian groups. As mentioned above, $\mathcal{Ab}_{\mathbb{Q}}$, which is the quotient by the category of bounded torsion objects, is an abelian tensor category. On the other hand, consider the category of vector spaces over $\mathbb{Q}$, $\mathcal{Vect}_{\mathbb{Q}}$, which is in fact the quotient of $\mathcal{Ab}$ with respect to the Serre subcategory of objects whose elements are torsion. In this sense one can say, that $\mathcal{Ab}_{\mathbb{Q}}$ is "bigger" then $\mathcal{Vect}_{\mathbb{Q}}$. And the natural functor

$$\begin{aligned} \mathcal{Ab}_{\mathbb{Q}} &\to \mathcal{Vect}_{\mathbb{Q}} \\ X_{\mathbb{Q}} \mapsto X \otimes \mathbb{Q} \end{aligned}$$

is not an equivalence of categories. Also, $\mathcal{Ab}_{\mathbb{Q}}$ does not have infinites sums and products.

$\mathcal{Ab}_{\mathbb{Q}}$ has homological dimension 1. For all $i \in \mathbb{N}$ and $X_{\mathbb{Q}}, Y_{\mathbb{Q}} \in \mathcal{Ab}_{\mathbb{Q}}$, we have

$$\mathrm{Ext}^i(X_{\mathbb{Q}}, Y_{\mathbb{Q}}) = \mathrm{Ext}^i(X, Y) \otimes \mathbb{Q}$$

but Ext^1 is exact in $\mathcal{Ab}_{\mathbb{Q}}$.

3 Spectra

Here we want to explain a more classical approach of the Eilenberg–MacLane functor. We start by explaining the main players. See for example [2].

The category of spectra is in a sense between spaces and abelian groups. This is where for example cohomology theories live, but also K-theory — as we all know important invariants in arithmetic geometry. The theory provides an access to such invariants, in that it assembles this kind of data in form of invariants with values in abelian groups in a way such that as a category it carries the same structure as the category of spaces, and therefore allows for a similar machinery.

3.1 Definitions

Definition 3.1.1. We call spectrum a sequence of simplicial sets

$$E = \{E^0, E^1, \ldots\}$$

together with structure maps

$$S^1 \wedge E^k \to E^{k+1}$$

for all $k \in \mathbb{N}_0$. A map of spectra $f : E \to F$ is a collections of maps of simplicial sets $f^k : E^k \to F^k$ for each k, that commute with the structure maps. We denote by $\mathcal{Sp}$ the category of spectra.

Some important examples are

Examples 3.1.2. 1. The sphere spectrum

$$\underline{S} = \{S^0, S^1, S^2, \dots, S^k = S^1 \wedge \cdots \wedge S^1\}$$

with identity as structure maps.

2. The Eilenberg–MacLane spectrum

$$H\,\mathbb{Z} = \{\widetilde{\mathbb{Z}}[S^k]\}_k$$

where $\widetilde{\mathbb{Z}}[X]$ for a pointed set X is the pointed set $\mathbb{Z}[X]/\,\mathbb{Z}[*]$. The natural map

$$\widetilde{\mathbb{Z}}[X] \wedge Y \to \widetilde{\mathbb{Z}}[X \wedge Y]$$

induces the structure maps.

The Eilenberg–MacLane spectrum is an example of an Ω-spectrum. That is, the loop and suspension functor are "almost" adjoint.

Definition 3.1.3. A spectrum is called Ω-spectrum if the adjoint of the structure ma gives rise to equivalences $E^k \to \Omega E^k$.

Although many of the important examples are Ω-spectra, it is for technical reasons easier to admit all spectra.

To relate spectra and simplicial sets, let X be a pointed simplicial set and E a spectrum. Then we get a new spectrum $E \wedge X = \{E^n \wedge X\}$. There are adjoint functors

$$\mathcal{S}p \rightleftharpoons \mathcal{S}$$

where the functor form right to left is

$$\sum^{\infty} X = \{S^n X\}$$

the suspension spectrum and the functor from left to right is

$$RE = E^0$$

the zeroth space. wedge The relevant equivalences, that give the right correspondence between cohomology and spectra are the stable equivalences.

Definition 3.1.4. Let E be a spectrum. The homotopy groups of E are defined to be

$$\pi_q E = \varinjlim \pi_{q+k} E^k$$

taken over the maps $\pi_{q+k} E^k \to \pi_{q+k} \Omega E^{k+1} \cong \pi_{q+k+1} E^{K+1}$.

This defines a functor from spectra to graded abelian groups.

Definition 3.1.5. Now we say that a map of spectra $f : E \to F$ is a stable equivalence, if it induces isomorphisms on the homotopy groups.

Example 3.1.6. If E and F are two spectra, we define $E \vee F = \{E^k \vee F^k\}$ with structure map $S^1 \wedge (E \vee F)^k \cong (S^1 \wedge E^k) \vee (S^1 \wedge F^k) \to E^{k+1} \vee F^{k+1}$ adn similar for $E \times F$. For two spectra, the natural map

$$E \vee F \to E \times F$$

is a stable equivalence. One has to show that the quotient is homotopy equivalent to the zero spectrum. This can be done on simplicial sets, and for those we have if $\pi_i X = 0$ for $i < n$ and $\pi_j Y = 0$ for $j < m$ then $\pi_k(X \wedge Y) = 0$ for $k < m + n$.

Now we pass to the homotopy category of spectra by formally inverting all stable equivalences. We denote it again by $\mathcal{Sp}$. This is often called the stable homotopy category.

Lemma 3.1.7. *The ∞-category $\mathcal{Sp}$ is stable.*

Proof. Lurie shows in [3, Cor. 1.4.2.17], that if an ∞ category admits finite limits, its category of spectrum objects is stable. □

(Stability means, roughly that the suspension is invertible.)

3.2 The t-structure

The category of spectra has a t-structure.

Definition 3.2.1. An element in $\mathcal{Sp}$ is called n-connective, if $\pi_i(X)$ is trivial for $i < n$. It is called connective, if it is 0-connective. It is connected, if it is 1-connective. We denote them by $\mathcal{Sp}_{\geqslant n}$

Likewise, an element is called n-truncated, if $\pi_(X)$ is trivial for $i > n$. It is discrete, if it is 0-truncated.

The t-structure is given by $\mathcal{Sp}_{\geqslant 0}$, the connective spectra, and $\mathcal{Sp}_{\leqslant 0}$, the discrete spectra.

Lemma 3.2.2. *We have $\mathcal{Sp}_{\geqslant n} = \mathcal{Sp}_{\geqslant 0}[-n]$.*

Proof. This is obvious from the definitions. □

We call the elements of $\mathcal{Sp}^- := \bigcup \mathcal{Sp}_{\geqslant n}$ eventually connective spectra. This is a stable ∞-subcategory of $\mathcal{Sp}$, and a tensor subcategory.

The t-structure homology functor is the homotopy groups functor. And the resulting truncations

$$X \mapsto \tau_{\leqslant n} X$$

are Postnikov truncations.

Proposition 3.2.3. *The t-structure on $\mathcal{Sp}$ is left and right complete, and its heart is canonically equivalent to the category of abelian groups.*

Proof. This is shown in [3, Prop. 1.4.3.6]. □

Lemma 3.2.4. *The t-structure is non-degenerate.*

Proof. We have to show that if all $\pi_n(X) = 0$ then $X = 0$. This follows after passing to the homotopy category. □

3.3 The tensor structure

The smash product $\wedge$, taken degree wise, gives a tensor structure on $\mathcal{Sp}$. And in fact, it is a symmetric tensor infinity category. The unit object is the sphere spectrum, and the necessary diagrams can be shown to commute using general properties as described in [3, 6.3.2]. On the heart $\mathcal{Ab}$ it induces the usual tensor product. It is right t-exact.

3.4 Relation to simplicial sets

We said, that spectra is the category where (co)homology lives. and in fact does a spectrum give rise to a (co)homology theory. For a simplicial set X we set

$$E_n(X) = \pi_n(E \wedge X) \qquad \text{and} \qquad E^n(X) = \pi_{-n}(E^X)$$

where E^X is given by the pointed simplicial maps from $X \wedge \delta[q]$ to E. A special case of this is the stable homotopy groups of a pointed space, which is given by taking the sphere spectrum for E

$$\pi_n^S(X) = \pi_n(\underline{S} \wedge X) = \varinjlim \pi_{n+k}(S^k \wedge X).$$

Another example is the (co)homology associated to the Eilenberg–MacLane spectrum, which gives the reduces simlicial (co)homology groups of a simplicial set X.

3.5 The Eilenberg–MacLane functor

In light of this, one might ask, if there is a connection between chain complexes and spectra. And indeed, if one considers a chain complex C, its information is retained by a sequence of "shifted" (non-negatively graded) complexes $C^0, C^1, C^2, \dots$

$$\begin{array}{cccc}
\vdots & \vdots & \vdots & \\
\downarrow{\scriptstyle d} & \downarrow{\scriptstyle d} & \downarrow{\scriptstyle d} & \\
C_2 & C_1 & C_3 & \dots \\
\downarrow{\scriptstyle d} & \downarrow{\scriptstyle d} & \downarrow{\scriptstyle d} & \\
C_1 & C_0 & C_{-1} & \dots \\
\downarrow{\scriptstyle d} & \downarrow{\scriptstyle d} & \downarrow{\scriptstyle d} & \\
C_0 & C_{-1} & C_{-2} & \dots
\end{array}$$

with isomorphisms $C^i_j \cong C^{i+1}_{j+1}$ and homology can be given by limits

$$\mathrm{H}_j(C) \cong \varinjlim \mathrm{H}_{n+j}(C^n)$$

The maps $C^i_j \cong C^{i+1}_{j+1}$ can be reformulated via a map

$$\mathbb{Z}[1] \otimes C^i \to C^{i+1}$$

which looks like the structure map.

So we can mimic the definition of the stable homotopy category to get spectrum objects of further categories.

1. The spectra of abelian groups $\mathcal{Sp}(\mathcal{Ab})$ with simplicial abelian groups, degree wise tensor and $\mathbb{Z}[S^1]$
2. The spectra of positive chain complexes $\mathcal{Sp}(D(\mathcal{Ab})^{\geqslant 0})$ with chain complexes concentrated in non-negative degree, tensor of chain complexes, and $\mathbb{Z}[1] = C^{norm}(\widetilde{\mathbb{Z}}[S^1])$. This identity is induced by the Dold-Kan equivalence.

   Theorem 3.5.1. *The normalised chain complex gives an equivalence of categories*

   $$C^{norm} : \mathcal{Ab} \to D(\mathcal{Ab})^{\geqslant 0}$$

   between simplicial abelian groups and positive concentrated chain complexes.
3. The spectra of chain complexes (as above but with $D(\mathcal{Ab})$ instead of $D(\mathcal{Ab})^{\geqslant 0}$.

The last category plays the role of chain complexes. We relate chain complexes and spectra in the following way.

$$D(\mathcal{Ab}) \underset{R}{\rightleftarrows} \mathcal{Sp}(D(\mathcal{Ab})) \underset{\text{include}}{\overset{\text{truncate}}{\rightleftarrows}} \mathcal{Sp}(D(\mathcal{Ab})^{\geqslant 0}) \underset{C^{norm}}{\rightleftarrows} \mathcal{Sp}(\mathcal{Ab}) \overset{\widetilde{\mathbb{Z}}[]}{\rightleftarrows} \mathcal{Sp}$$

where everything left to $\mathcal{Sp}(\mathcal{Ab})$ is an equivalence on the associated homotopy categories.

This is a very rough sketch of the Eilenberg–MacLane functor. We denote it by EM It is t-exact and on the heart it equals the identity functor. Moreover, it sends rings to rings and modules to modules. It is naturally an ∞-category functor.

Consider now the Eilenberg–MacLane spectrum again:

$$H\,\mathbb{Z} = \{\widetilde{\mathbb{Z}}[S^k]\}_k$$

. This is in fact the image of $\mathbb{Z}$ under the Eilenberg–MacLane functor and as such is a unital ring spectrum. We denote it by $\mathbb{Z}_{\mathcal{S}p}$.

The Eilenberg–MacLane functor lifts to a functor

$$D(\mathcal{A}b) \to \mathbb{Z}_{\mathcal{S}p} - \mathrm{mod}$$

and this is in fact an equivalence of categories.(I couldn't find the reference for this in Luris's book.)

3.6 Quasi-isogenies

We consider now again $\mathcal{S}p^-$, the category of eventually connective spectra. A map in $\mathcal{S}p^-$ is called a quasi-isogeny if all induced maps on the homotopy groups

$$\pi_n(X) \to \pi_n(Y)$$

are isogenies.

Lemma 3.6.1. *This is equivalent to the condition that all maps induced on truncations*

$$\tau_{\leqslant n} X \to \tau_{\leqslant n} Y$$

are isogenies in the category of spectra.

Proof. This follows from the definition of the t-structure. □

Example 3.6.2. $X \in \mathcal{S}p^-$ is qusi-isogenous to 0 if all $\pi_n(X)$ are isogenous to 0, and we have seen, that this means, they are bounded torsion. Equivalentyl, this means that all $\tau_{\leqslant n} X$ are bounded torsion spectra.

Remark 3.6.3. Thus to consider $\mathcal{S}p^-$ up to quasi-isogeny is roughly the same as to consider $\mathcal{S}p$ up to isogeny.

Lemma 3.6.4. *The category of spectra quasi-isogenous to 0 form a thick subcategory of $\mathcal{S}p^-$.*

Proof. They are clearly closed under extensions. □

Lemma 3.6.5. *This subcategory is an ideal in $\mathcal{S}p^-$ for the smash product.*

Proof. Indeed, let $X, Y \in \mathcal{S}p^-$ and Y quasi-isogenous to 0. We have to show that $X \wedge Y$ is again quasi-isogenous to 0. we can shift the objects, so that slog we can assume that they are in fact connective. As we take the smash product component (degree) wise, we have $\tau_{\leqslant n}(X \wedge Y) = \tau_{\leqslant n}(X \wedge \tau_{\leqslant n} Y)$. But $\tau_{\leqslant n} Y$ is isogenous to 0 for all n, meaning, there is a k such that $k\,\mathrm{id}_{\tau_{\leqslant n} Y} = 0$. This same k will therefore kill $\mathrm{id}_{\tau_{\leqslant n}(X \wedge Y)}$. □

Now we are in a position to take the Verdier quotient: Let $\mathcal{S}p^-_{\mathbb{Q}}$ be the quotient of $\mathcal{S}p^-$ by this ideal. This is again a symmetric tensor t-category. With reference to abodes remark, we call this category the category of spectra up to quasi-isogeny.

We can do the same for the derived category $D^-(\mathcal{A}b)$ of bounded above chain complexes and get the category of bounded chain complexes up to bounded torsion homology. We denote it by $D^-(\mathcal{A}b_{\mathbb{Q}})$.

They are again related via the Eilenberg–MacLane functor. It sends $\mathcal{S}p^-$ to $D^-(\mathcal{A}b)$ and passing to quotients, we get a t-exact functor

$$D^-(\mathcal{A}b_{\mathbb{Q}}) \to \mathcal{S}p^-_{\mathbb{Q}}\,.$$

Proposition 3.6.6. *This is an equivalence of tensor triangulated categories.*

Proof. Recall that the functor

$$D(\mathcal{Ab}) \to \mathbb{Z}_{\mathcal{S}p} -mod$$

induced by the Eilenberg–MacLane functor is an equivalence of categories, where $\mathbb{Z}_{\mathcal{S}p}$ is the Eilenberg–MacLane spectrum. We look now at the category $\mathbb{Z}_{\mathcal{S}p} -mod^-$ of eventually connective $\mathbb{Z}_{\mathcal{S}p}$-modules and its quotient $\mathbb{Z}_{\mathcal{S}p} -mod^-_{\mathbb{Q}}$ by objects quasi-isogenous to 0.

We have the forgetful functor

$$\mathscr{U} : \mathbb{Z}_{\mathcal{S}p} -mod^- \to \mathcal{S}p^-$$

with left adjoint

$$\begin{aligned} \mathbb{Z}_{\mathcal{S}p} \wedge : \mathcal{S}p^- &\to \mathbb{Z}_{\mathcal{S}p} -mod^- \\ X &\mapsto \mathbb{Z}_{\mathcal{S}p} \wedge X \end{aligned}$$

We pass to quotients and get a pair of adjoint functors

$$\mathbb{Z}_{\mathcal{S}p} -mod^-_{\mathbb{Q}} \underset{\mathbb{Z}_{\mathcal{S}p} \wedge}{\overset{\mathscr{U}}{\rightleftarrows}} \mathcal{S}p^-_{\mathbb{Q}}$$

Considering the equivalence of categories between $\mathbb{Z}_{\mathcal{S}p} -mod$ and $D(\mathcal{Ab})$, one only needs to show that these are inverse to each other.

For $X \in \mathcal{S}p^-$ and $Y \in \mathbb{Z}_{\mathcal{S}p} -mod^-$ we have to show that the adjunction maps

$$\begin{aligned} \alpha_X : X &\to \mathbb{Z}_{\mathcal{S}p} \wedge X \\ \alpha^\vee_Y : \mathbb{Z}_{\mathcal{S}p} \wedge Y &\to Y \end{aligned}$$

are quasi-isogenies.

First we notice that $\alpha_X = \alpha_S \wedge \mathrm{id}_X$ where S is the sphere spectrum (as unit object, $\alpha_S : S \to \mathbb{Z}_{\mathcal{S}p}$ the unit map). The by α_S induced map on π_0 is $\mathrm{id}_{\mathbb{Z}} = \pi_0(\alpha_S)$, in particular $\pi_0(\mathrm{Cone}(\alpha_S)$ is trivial, and thus the higher homotopy groups of this cone are all finite. As we are only interested in α_X modulo bounded torsion, this tells us that $\mathrm{Cone}(\alpha_S)$ is quasi-isogenous to 0.Consequently,

$$\mathrm{Cone}(\alpha_X) = \mathrm{Cone}(\alpha_S) \wedge X$$

is quasi-isogenous to 0 as well. (This can be shown similarly as in the above lemma.) This implies that α_X is a quasi-isogeny.

Now we come to $\alpha^\vee_Y$. We have $\alpha^\vee_Y \circ \alpha_Y = \mathrm{id}_Y$, and we know by the previous paragraph, that α_Y is a quasi-isogeny. Thus there is a morphism g such that $\alpha_Y \circ g = n\,\mathrm{id}$ and $g \circ \alpha_Y = n\,\mathrm{id}$ for some n. So we have

$$n\alpha^\vee_Y = \alpha^\vee_Y n \alpha_Y g = ng$$

So if $f := n\alpha_Y$

$$f\alpha^\vee_Y = \alpha_Y n \alpha^\vee_Y = \alpha_Y ng = n^2\,\mathrm{id}$$

and

$$\alpha^\vee_Y f = \alpha^\vee_Y n \alpha_Y = ng\alpha_Y = n^2\,\mathrm{id}$$

(this needs some more details). □

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