Goodwillie's theorem

Preliminaries to understand the work of Beilinson on
Relative continuous K-theory and cyclic homology
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Abstract

These are the notes of my talk given in the Oberseminar "Continuous K-theory of p-adic rings" organised by Uwe Jannsen, Moritz Kerz, Guido Kings and Niko Naumann at the University of Ratisbon during the summer term 2014. The goal was to understand and summarise the proof of Goodwillie's theorem.

Résumé

Voilà les notes pour mon exposé dans l'Oberseminar "Continuous K-theory of p-adic rings" organisé par Uwe Jannsen, Moritz Kerz, Guido Kings et Niko Naumann à l'Université de Ratisbonne en semestre d'été 2014. Le but etait de comprendre et de rassembler le matériel pour la démontration du théorème de Goodwillie.

Key Words : Cyclic homology,Hochschild homology, Loday–Quillen–Tsygan theorem, Lie algebra homology, algebraic K-theory, Goodwilie's theorem, Lazard isomorphism.
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Introduction

The program of the seminar is given here: `http://www.mathematik.uni-regensburg.de/kerz/ss14/oberseminar.pdf`

In this weekly Oberseminar we studied cylic homology, algebraic K-theory and the Lazard isomorphism. The idea is to learn topics, interesting in their own, which form the background for understanding Beilinson's recent paper [1]. We will study his paper in detail in form of a workshop taking place September 29 until October 2.

Cyclic homology was originally introduced by Connes and Tsygan as a form of homology of algebras generalizing in some sense de Rham cohomology of smooth commutative algebras. Roughly speaking it is a 'simple' homology theory which sometimes allows one to calculate more complicated homology theories explicitly. One key observation is the theorem of Loday–Quillen–Tsygan which calculates the Lie algebra homology of the general linear group $\mathfrak{gl}(A) = \varinjlim \mathfrak{gl}_r(A)$ of a ring A in terms of cyclic homology. Another important result is Goodwillie's calculation of algebraic K-theory relative to a nilpotent ideal in characteristic zero in terms of cyclic homology. We will study both results in the seminar. For Goodwillie we need some background from algebraic K -theory, in particular Quillen's +-construction and the Volodin construction. Variants of all these techniques are used by Beilinson in [1]. In a second part of the seminar, logically independent from the first, we study the Lazard isomorphism, which is another key ingredient of [1].

The weekly seminar was rounded up by a weeklong workshop in September 2014 in Kleinwalsertal. Here Beilinson's article was studied in depth and discussions evolved around further developements.

1 Goodwillie's theorem

The goal of this section is to prepare and present the theorem of Goodwillie making precise the relation between algebraic K-theory and cyclic homology of nilpotent ideals. The main reference is [2], but we also refer to [4] and [5].

1.1 Statement of the theorem

The main theorem of this section computes the relative K-groups in terms of relative cyclic homology groups for nilpotent ideals in characteristic zero.

Theorem 1.1.1. ***(T. Goodwillie)*** *Let A be a ring and let I be a two-sided nilpotent ideal. For all $n \in \mathbb{N}$ there is a canonical isomorphism*

$$\rho : K_n(A, I) \otimes \mathbb{Q} \xrightarrow{\sim} HC_{n-1}(A, I) \otimes \mathbb{Q} .$$

Remark 1.1.2. This should correspond to the Goodwillie Chern character but we didn't check this claim (for more details see [2]).

1.2 Overview of the proof

The steps of the proof are the following. First one notes that the general case can be reduced to the case where A is a $\mathbb{Q}$-algebra and I a nilpotent ideal. Then the proof of this rational case is devided in three steps.

The first part looks at the K-theory side. Here a relative Volodin construction $X(A, I)$ provides a model for relative algebraic K-theory in the sense that

$$K_*(A, I) \cong \operatorname{Prim} \mathrm{H}_* \left(X(A, I) \right) ,$$

where Prim denotes the primitive elements of a Hopf algebra.

The second part is concerned with the Lie side and constructs a relative Volodin complex $x(A, I)$ which provides a model for relative cyclic homology in terms of an isomorphism

$$HC_{*-1}(A, I) \cong \operatorname{Prim} / h_* \left(x(A, I) \right) .$$

The third part compares the the right hand sides of both isomorphisms. The theory of Malcev provides a map of Hopf algebras

$$\mathrm{H}_* \left(X(A, I), \mathbb{Q} \right) \to \mathrm{H}_* \left(x(A, I) \right) ,$$

which turns out to be an isomorphism. Restricting to the primitive part on both sides and tensoring with $\mathbb{Q}$ yield the proof of the theorem.

1.3 The K-theory side

This part has been presented in detail by Uli Bunke in the previous talk. I just want to recall some facts that carry over to the Lie theory side.

We have seen, that the relative Volodin space $X(A, I)$ can be constructed as a subset of a simplicial space. Recall that the absolute Volodin space $X(A)$ is acyclic and provides a model for $\mathrm{BGL}(A)^+$. While the relative Volodin space is not acyclic, it provides a model for K-theory after applying the $+$-construction to it, in the sense that

$$X(A, I)^+ \to \mathrm{BGL}(A)^+ \to \overline{\mathrm{BGL}}(A/I)^+$$

is a homotopy fibration. This gives a spectral sequence (the homology spectral sequence of a fibration)

$$E^2_{pq} = \mathrm{H}_q(\overline{\mathrm{BGL}}(A/I)^+) \otimes \mathrm{H}_q(X(A, I)^+) \quad \Rightarrow \quad \mathrm{H}_{p+q}(\mathrm{BGL}(A)^+)$$

We want to mimick this on the Lie theory side.

Remark 1.3.1. This is possible because for a nilpotent Lie algebra $\mathfrak{G}$ and correspoinding nilpotent group G we have the identification

$$\mathrm{H}_*(\mathrm{B}\,G, \mathbb{Q}) \cong \mathrm{H}_*(\mathfrak{G})$$

In fact, the left hand side can be computed by the Eilenberg–MacLane complex

$$C_n(G) = k[G]^n$$

together with the boundary map

$$d(g_0, \ldots, g_n) = (g_0g_1, g_2, \ldots, g_n) - (g_0, g_1g_2, \ldots, g_n) + - \cdots (-1)^n(g_0, \ldots, g_{n-1}g_n)$$

which reminds us of the Eilenberg–Chevalley complex for Lie algebras.

1.4 The Lie side (rational case)

Bearing the above remark in mind, we make a Volodin type constructions on Lie algebras. For an ordering γ of $\{1, \ldots, n\}$

$$t^\gamma_n(A, I) := \left\{ (a_{ij}) \in \mathfrak{Gl}_n(A) \quad \middle| \quad a_{ij} \in I \text{ if } i \overset{\gamma}{\not<} j \right\}$$

This is of course a nilpotent Lie algebra.

Now we take the Chevalley–Eilenberg complex

$$C_*(t^\gamma_n(A, I)) \hookrightarrow C_*(\mathfrak{Gl}(A))$$

where for a nilpotent Lie algebra $\mathfrak{G}$

$$C_n(\mathfrak{G}) = \bigwedge^n \mathfrak{G}$$

with boundary

$$d(g_1 \wedge \ldots \wedge g_n) = \sum_{1 \leqslant i < j \leqslant n} (-1)^{i+j}[x_i, x_j] \wedge x_1 \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n.$$

Now set

$$x_n(A, I) = \sum_\gamma C_*(t^\gamma_n(A, I)$$

and take the colimit over all n

$$x(A, I) = \operatorname{colim} x_n(A, I)$$

Analogous to $X(A)$ we have a here non-relative construction $x(A)$ using

$$t^\gamma_n(A) = \left\{ (a_{ij}) \in \mathfrak{Gl}_n(A) \quad \middle| \quad a_{ij} = 0 \text{ if } i \overset{\gamma}{\not<} \right\}$$

and then procede as above.

Theorem 1.4.1 (Suslin–Wodzicki). *The Volodin complex $x(A)$ is acyclic in positive degree (just as $X(A)$).*

But the analogies continue. We have as well a spectral sequence.

$$E^2_{pq} = \mathrm{H}_p\left(\mathfrak{Gl}(A/I)\right) \otimes \mathrm{H}_q\left(x(A,I)\right) \quad \Rightarrow \quad \mathrm{H}_{p+q}\left(\mathfrak{Gl}(A)\right)$$

(maybe modify an argument of Eckmann–Stammbeck). If we take primitive elements in this spectral sequance, we will see, that kills a lot of elements, wich is very helpful for the following lemma.

Lemma 1.4.2. *There is a long exact sequence*

$$\cdots \operatorname{Prim} \mathrm{H}_{n+1}(\mathfrak{Gl}(A)) \to \operatorname{Prim} \mathrm{H}_{n+1}(\mathfrak{Gl}(A/I)) \to \operatorname{Prim} \mathrm{H}_n(x(A,I)) \to \cdots$$

Proof. Taking primitive elements in the above spectral sequence leasves us with

$$E^2_{pq} = \begin{cases} \operatorname{Prim} \mathrm{H}_q(x(A,I)) & \text{if } p = 0 \\ \operatorname{Prim} \mathrm{H}_p(\mathfrak{Gl}(A/I) & \text{if } q = 0 \\ 0 & \text{if} p > 0 \text{ and } q > 0 \end{cases}$$

With all the vanishing entries, a general theorem of spectral sequences gives us

$$\cdots \to E^2_{0n} \to E_n \to E^2_{n0} \xrightarrow{d^n} E^2_{0,n-1} \to E_{n-1} \to \cdots$$

Taking into account what the E-terms stand for, gives us the desired long exact sequence. □

On the oter hand, we have the long exact sequence of cyclic homology.

$$\cdots \to HC_n(A) \to HC_n(A/I) \to HC_{n-1}(A,I) \to \cdots$$

Moreover, we want to use the Quillen–Loday–Tsygan theoremm that gives us for A and A/I respectively, isomorphisms

$$\rho : \operatorname{Prim} \mathrm{H}_*(\mathfrak{Gl}(A)) \xrightarrow{\sim} HC_{*-1}(A) \tag{1.4.1}$$

$$\rho : \operatorname{Prim} \mathrm{H}_*(\mathfrak{Gl}(A/I)) \xrightarrow{\sim} HC_{*-1}(A/I) \tag{1.4.2}$$

We need to show that ρ sends $x(A/I)$ to $HC(A,I)$. So lets look at its construction.

There is a map

$$\begin{aligned} \theta : \bigwedge^{n+1} \mathfrak{Gl}_r(A) &\to C^\lambda_n(\mathscr{M}_r(A)) \\ \alpha_0 \wedge \ldots \wedge \alpha_n &\mapsto \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)(\alpha_0, \alpha_{\sigma(1)}, \ldots, \alpha_{\sigma(n)}) \end{aligned}$$

where C^λ denotes the Eilenberg–MacLane complex modulo the cyclic operator. This induces by functoriality a map on homology

$$\theta_* : \mathrm{H}_{n+1}(\mathfrak{Gl}_r(A)) \to \mathrm{H}_n(C^\lambda(\mathscr{M}_r(A))).$$

Then we apply the trace map to get

$$\operatorname{tr}_* \circ \theta_* : \mathrm{H}_*(\mathfrak{Gl}_r(A)) \to \mathrm{H}_{*-1}(A) = \mathrm{H}_*(C^\lambda(A)).$$

Passing to the limit over r gives the map ρ. Now if one traces the map, one can see that this map just constructed sends the complex $x(A,I)$ indeed to $\ker(C^\lambda(A) \to C^\lambda(A(I))$. Consequently, ρ maps the short exact sequences to each other

$$\begin{array}{ccccccccc} \cdots \longrightarrow & \operatorname{Prim} \mathrm{H}_{n+1}(\mathfrak{Gl}(A)) & \longrightarrow & \operatorname{Prim} \mathrm{H}_{n+1}(\mathfrak{Gl}(A/I)) & \longrightarrow & \operatorname{Prim} \mathrm{H}_n(x(A/I)) & \longrightarrow \cdots \\ & \downarrow \sim & & \downarrow \sim & & \downarrow & \\ \cdots \longrightarrow & HC_n(A) & \longrightarrow & HC_n(A/I) & \longrightarrow & HC_{n-1}(A,I) & \longrightarrow \cdots \end{array}$$

The Five Lemma together with isomorphisms **??** gives the desired isomorphism

$$HC_{*-1}(A,I) \cong \operatorname{Prim}/h_*\left(x(A,I)\right).$$

1.5 Connecting them via Malcev theory (rationally)

We are now ready to connect the two parts. We have seen, that the Volodin constructions on both sides are very similar. On the K-theory side, we have nilpotent groups $T^\gamma(A,I)$ and on the Lie side we have nilpotent Lie algebras $t^\gamma(A,I)$. These are made to correspond to each other. In fact, the exponential map relates Lie algebras and Lie groups. In particular it gives a one-to-one correspondence of nilpotent Lie algebras over $\mathbb{Q}$ and nilpotent groups.

(The map is constructed as follows: Let $\mathfrak{n}$ be a nilpotent Lie $\mathbb{Q}$-algebra and J the augmentation ideal of it universal enveloping algebra $U(\mathfrak{n})$. Complete $U(\mathfrak{n})$ J-adically. Then the exponential $\exp : \widehat{J} \to 1 + \widehat{J}$ (I don't know what this means should it not be $\exp : \widehat{U(\mathfrak{n})} \to 1 + \widehat{U(\mathfrak{n})}$?) map is convergent J-adically. Then the image of $\mathfrak{n}$ under the expnential map yields a nilpotent group

$$N := \exp(\mathfrak{n}) \subset (1 + \widehat{J})^\times,$$

and the assignement $\mathfrak{n} \mapsto N$ is a functor from the category of nilpotent Lie algebras to nilpotent groups.)

Under this map we see clearly that $t^\gamma(A,I)$ corresponds to $T^\gamma(A,I)$.

To a nilpotent Lie algebra, we have already associtated the Chevalley–Eilenberg complex $C_*(\mathfrak{N})$. Analogously we can associate to N (as a discrete group) the Eilenberg–McLane complex $C_*(N)$ (take any complex that calculates group homology). The next step, where Malcev theory comes in fact in, is to show that these complexes are quasi-isomorphic.

Proposition 1.5.1. *For any nilpotent Lie $\mathbb{Q}$-algebra $\mathfrak{n}$ thre is a natural diagram of complexes*

$$C_*(N) \to C_*(N, \mathfrak{n}) \leftarrow C_*(\mathfrak{n})$$

where the two maps are quasi-isomorphisms.

Proof. The complexes $C_*(N)$ and $C_*(\mathfrak{n})$ computes the homology of a nilpotent group and a nilpotent Lie algebra respectively, which can also be computed in different ways. We will use this fact.

Consider therefore on the one hand the group ring $\mathbb{Q}[N]$, together with the standard complex $C_*^{st}(\mathbb{Q}[N])$ that computes the Tor-complex $\mathrm{Tor}^{\mathbb{Q}[N]}(\mathbb{Q},\mathbb{Q})$. We have

$$\mathrm{H}_*(N,\mathbb{Q}) = \mathrm{H}_*(C_*(N)) = \mathrm{H}_*(\mathrm{Tor}^{\mathbb{Q}[N]}(\mathbb{Q},\mathbb{Q})).$$

On the other hand, we have seen that there Lie algebra homology can be calculated with the Tor-functor as well, using the universal enveloping algebra. Thus set $C_*^{st}(U(\mathfrak{n}))$ the standard complex, which computes $\mathrm{Tor}^{U(\mathfrak{n})}(\mathbb{Q},\mathbb{Q})$. We have

$$\mathrm{H}_*(\mathfrak{n},\mathbb{Q}) = \mathrm{H}_*(C_*(\mathfrak{n})) = \mathrm{H}_*(\mathrm{Tor}^{U(\mathfrak{n})}(\mathbb{Q},\mathbb{Q}).$$

We compare these two standard complexes.

First complete $\mathbb{Q}[N]$ and $U(\mathfrak{n})$ with respect to the augmentation ideal and get $\widehat{\mathbb{Q}[N]}$ and $\widehat{U(\mathfrak{n})}$. By definition there is an inclusion $N \hookrightarrow \widehat{U(\mathfrak{n})}$, which can be scalar extended to th group algebra over $\mathbb{Q}$ and then completed to

$$\widehat{\mathbb{Q}[N]} \to \widehat{U(\mathfrak{n})}.$$

When $\mathfrak{n}$ is finitely generated, we have the following properties:

1. (by Malcev) the algebra morphism $\widehat{\mathbb{Q}[N]} \to \widehat{U(\mathfrak{n})}$ is an isomorphism of Hopf algebras.
2. $\widehat{\mathbb{Q}[N]}$ is flat as a $\mathbb{Q}[N]$-algebra.
3. $\widehat{U(\mathfrak{n})}$ is flat as a $U(\mathfrak{n})$-algebra.

So in the case of a finitely generated nilpotent Lie $\mathbb{Q}$-algebra $\mathfrak{n}$ we get

1. Form the first property, because Tor is functorial,

$$C_*^{st}(\widehat{U(\mathfrak{n})}) \cong C_*^{st}(\widehat{\mathbb{Q}[N]}).$$

2. From the second that
$$C_*^{st}(\mathbb{Q}[N]) \to C_*^{st}(\widehat{\mathbb{Q}[N]})$$
is quasi-isomorphic.

3. And from the third one that
$$C_*^{st}(U(\mathfrak{n})) \to C_*^{st}(\widehat{U(\mathfrak{n})})$$
is quasi-isomorphic.

Putting all which was said together then gives a diagram

$$C_*(N) \to C_*(N, \mathfrak{n}) \leftarrow C_*(\mathfrak{n}),$$

where the middle complex is $C_*^{st}(\widehat{U(\mathfrak{n})}) \cong C_*^{st}(\widehat{\mathbb{Q}[N]})$, and where the maps are quasi-isomorphisms.

Now since every nilpotent Lie $\mathbb{Q}$-algebra is inductive limit of its finitely generated sub-Lie algebras, and the functors involved commute with taking inductive limit, we can pass to the general case. □

It follows that there is a natural isomorphism of cohomology

$$\mathrm{H}_*(N, \mathbb{Q}) \cong \mathrm{H}_*(\mathfrak{n}, \mathbb{Q})$$

Since $t^\gamma(A, I)$ is nilpotent, and the associated Lie group is $T^\gamma(A, I)$, we apply the previous result to these in order to say something about the respevtive Volodin constructions.

Proposition 1.5.2. *There is a canonical quasi-isomorphism between $X(A, I)$ and $x(A, I)$.*

Proof. We have diagrams of quasi-isomorphisms as in the previous proposition for $N = T^\gamma(A, I)$ and $\mathfrak{n} = t^\gamma(A, I)$ for each gamma. This gives a functorial quasi.isomorphism

$$C_*(T_n^\gamma(A, I)) \to C_*(t_n^\gamma(A, I))$$

for each γ. Now to patch these together, fix n. By functorialty the above gives a map

$$\sum_\gamma C_*(T_n^\gamma(A, I)) \to \sum_\gamma C_*(t_n^\gamma(A, I))$$

and with Mayer-Vietoris this is again a quasi-isomorphism. As all constructions are compatible on each degree n, one can pass to the colimit and get

$$X(A, I) = \sum C_*(T^\gamma) \to \sum C_*(T^\gamma, t^\gamma) \leftarrow \sum C_*(t^\gamma) = x(A, I)$$

which is exactly what we hoped for. □

Corollary 1.5.3. *There is an isomorphism* $\mathrm{H}_*(X(A, I)) \cong \mathrm{H}_*(x(A, I))$.

1.6 Putting the pieces together

Let A be a $\mathbb{Q}$-algebra. Then the last three sections showed:

$$\begin{aligned} K_*(A, I)_{\mathbb{Q}} &\cong \operatorname{Prim} \mathrm{H}_*(X(A, I)) \\ \mathrm{H}_*(X(A, I)) &\cong \mathrm{H}_*(x, (A, I)) \\ \operatorname{Prim} \mathrm{H}_*(x(A, I)) &\cong HC_{*-1}(A, I) \end{aligned}$$

1.7 Reduction to the rational case

The claim of the main theorem is formulated for a ring A and the ground ring for cyclic homology and algebraic K-theory is $\mathbb{Z}$ (before both sides are tensored with $\mathbb{Q}$. However, there are localisation isomorphisms, that allow us to reduce the argumentation to the rational case, i.e. A is a $\mathbb{Q}$-algebra.

Let's first take a look at the Lie side. Let A be an asociative unital algebra over a commutative ring k (for example a unital ring (over $\mathbb{Z}$ in this case)). Recall that $HC_*(A)$ is the homology of the total complex of a cyclic bicomplex $\mathscr{C}(A)$ [3, Definition p. 568]. If $f : A \to A'$ is a morphism of k-algebras, it induces a map of bicomplexes, and therefore functorial maps

$$HC_n(A) \to HC_n(A').$$

In particular in the case of base change, that is, in the case of a sequence of ring morphisms $k \to K \to A$ there is a natural map of k-modules

$$HC_*^k(A) \to HC_*^K(A),$$

which is an isomorphism provided that K is a localisation of k (localisation being exact). In particular, if A is a $/QQ$-algebra

$$HC_*^{\mathbb{Z}}(A)_{\mathbb{Q}} \cong HC_*^{\mathbb{Q}}(A).$$

Similar for relative cyclic homology. So for a ring A, we obtain

$$HC_*^{\mathbb{Z}}(A, I)_{\mathbb{Q}} \cong HC_*^{\mathbb{Q}}(A_{\mathbb{Q}}, I_{\mathbb{Q}}).$$

The analogue for the K-theory side is a little more involved.

Proposition 1.7.1. *Let A be a ring and I a two-sided nilpotent ideal. Then there is a natural isomorphism*

$$K_*(A, I)_{\mathbb{Q}} \cong K_*(A_{\mathbb{Q}}, I_{\mathbb{Q}})_{\mathbb{Q}}.$$

Proof. As we have seen in the K-theory part, the relative K-groups appearing in the statement can be interpreted as primitive parts of certain Hopf algebras. Thus is suffices to prove the claim for those, i.e. for the functor $\mathrm{H}_*(X(\cdot, \cdot))$ coming from the Volodin construction. Recall that the Volodin space $X(A, I)$ is defined to be the union of classifying spaces

$$X(A, I) = \bigcup_{\gamma} BT^{\gamma}(A, I)$$

where the union is over all orderings γ of finite sets $\{1, \dots, n\}$, where n varies. For any ordering γ (and associated n) we have defined the triangular subgroup

$$T^{\gamma}(A, I) := \big\{1 + (a_{ij}) \in \mathbf{GL}_n(A) \quad \big| \quad a_{ij} \in I \text{ if } \not<^{\gamma} j\big\}$$

so that for triangular subgroups associated to two orderings γ_1 and γ_2 the union is again a triangular subgroup associated to the ordering

$$\gamma_3 : \quad i < j \text{ iff } i <^{\gamma_1} j \text{ and } i <^{\gamma_2} j.$$

Thus the classifying space of the triangular group $T^{\gamma_3}(A, I)$ classifies the intersection of the classifying spaces $BT^{\gamma_1}(A, I) \cap BT^{\gamma_2}(A, I)$. This allows us to use a Mayer-Vietoris argument. Indeed, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{H}_n(BT^{\gamma}(A, I) \cap \bigcup BT^{\gamma_1}(A, I))_{\mathbb{Q}} & \longrightarrow & \mathrm{H}_n(BT^{\gamma}(A, I))_{\mathbb{Q}} \oplus \mathrm{H}_n(\bigcup BT\gamma_1(A, I))_{\mathbb{Q}} & \longrightarrow & \mathrm{H}_n(X(A, I))_{\mathbb{Q}} & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & \\
\mathrm{H}_n(BT^{\gamma}(A_{\mathbb{Q}}, I_{\mathbb{Q}}) \cap \bigcup BT^{\gamma_1}(A_{\mathbb{Q}}, I_{\mathbb{Q}}))_{\mathbb{Q}} & \longrightarrow & \mathrm{H}_n(BT^{\gamma}(A_{\mathbb{Q}}, I_{\mathbb{Q}}))_{\mathbb{Q}} \oplus \mathrm{H}_n(\bigcup BT^{\gamma_1}(A_{\mathbb{Q}}, I_{\mathbb{Q}}))_{\mathbb{Q}} & \longrightarrow & \mathrm{H}_n(X(A_{\mathbb{Q}}, I_{\mathbb{Q}}))_{\mathbb{Q}} & \longrightarrow
\end{array}$$

where the union is over $\gamma_1 \neq \gamma$. If we obtain the isomorphism for $BT^\gamma(\cdot,\cdot)$ for arbitrary γ, the isomorphism for $X(\cdot,\cdot)$ follows from this diagram by induction.

We note that for a fixed order γ the triangular group $T^\gamma(A,I)$ is nilpotent because I is nilpotent.

Thus, fix an ordering γ and consider the group $T^\gamma(A,I)$. As this is basically a reordering of an upper triangular group, one can see that this is indeed a nilpotent group.

Lemma 1.7.2. *Let N be nilpotent. The we have canonical identifications of homology rings*

$$\mathrm{H}_*(N,\mathbb{Z})\otimes\mathbb{Q}\cong\mathrm{H}_*(N_\mathbb{Q},\mathbb{Z})\otimes\mathbb{Q}\cong\mathrm{H}_*(N_\mathbb{Q},\mathbb{Q}).$$

Proof. There is a spectral sequence

$$\mathrm{H}_p\left(N/[N,N],\mathrm{H}_q([N,N])\right) \quad\Rightarrow\quad \mathrm{H}_{p+q}(N)$$

hence we may assume that N is abelian. Moreover, assume that N is finitely generated (for general N pass to the limit over finitely generated subgroups). In this case, we can choos a presentation of the form

$$N\cong\mathbb{Z}^r\times T$$

where T is of torsion (and thus finite). In this case, the claim of the lemma is obvious. □

It follows with notations as before

$$\mathrm{H}_*(BT^\gamma(A,I))\otimes\mathbb{Q}\cong\mathrm{H}_*(T^\gamma(A,I),\mathbb{Z})\otimes\mathbb{Q}\cong\mathrm{H}_*(T^\gamma(A_\mathbb{Q},I_\mathbb{Q}),\mathbb{Q})\cong\mathrm{H}_*(BT^\gamma(A_\mathbb{Q},I_\mathbb{Q})\otimes\mathbb{Q}\,.$$

Taking colimits gives

$$\mathrm{H}_*(X(A,I))\otimes\mathbb{Q}\cong\mathrm{H}_*(X(A_\mathbb{Q},I_\mathbb{Q}))\otimes\mathbb{Q}$$

as wanted. □

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