A new approach to the de Rham-Witt complex after Bhatt–Lurie–Mathew

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In arithmetic and algebraic geometry cohomology theories provide important tools. One of the more well-known theories for (smooth) algebraic varieties X over the complex numbers is (algebraic) de Rham cohomology $\Omega^\bullet_{X/\mathbb{C}}$. However, over fields k of positive characteristic p, algebraic de Rham cohomology is a less satisfactory invariant. This is due in part to the fact that it takes values in the category of k-vector spaces, and therefore consists entirely of p-torsion.

To remedy this, Berthelot and Grothendieck introduced *crystalline cohomology* [1] for smooth projective varieties X over a perfect field k of characteristic $p > 0$ as sheaf cohomology of a certain site. These cohomology groups have coefficients in the Witt vectors $W(k)$ and after inverting p, they provide for smooth projective k-varieties a Weil cohomology with p-adic coefficients. Later Bloch and Illusie [6] gave another description of crystalline cohomology, which is closer in spirit to the definition of algebraic de Rham cohomology: they showed that the crystalline cohomology groups can be computed by a complex of sheaves on the étale site, the de Rham–Witt complex $W\Omega^\bullet_X/k$. However crystalline cohomology, and therefore the hypercohomology of the de Rham–Witt complex, do not work well for open or singular varieties over k [4].

In 2017, Bhatt, Lurie and Mathew introduced a new approach to the construction of the de Rham–Witt complex [5]. They called it "Constructing de Rham–Witt complexes on a budget". And indeed, it is simpler in the sense that it avoids lengthy calculations for standard affine spaces that appear in [6]. Moreover, it is potentially of greater interest, as for singular varieties it provides promising substitutes for crystalline cohomology.

In this seminar, we want to study this new approach to the de Rham–Witt complex, and depending on our time constraints consider further developments and (potential) applications. As a guideline we will use Illusie's report [7] on [5].

1 Context and overview (26 April 2023)

In this talk, we want to get some historical context to motivate our studies.

Give some background on the motivation and properties of crystalline cohomology and the de Rham–Witt complex. Recall the décalage of filtrations observed by Deligne for general complexes and explain how it reappears in the context of p-adic Hodge theory.

Explain how the décalage functor emerges and how this motivated the new construction of the de Rham–Witt complex.

Give an overview of the construction of Bhatt–Lurie–Mathew's approach to the de Rham–Witt complex and explain some of the differences to the original construction.

— [7, §1], [5, §1.1-§1.3]

2 Dieudonné complexes (7 June 2023)

In this talk, we start with basic definitions.

Define the category of saturated Dieudonné complexes and (saturated) Dieudonné complexes and discuss the saturation functor. Give an example.

Define the completion of a saturated Dieudonné complex and show its universal property. Discuss the relation between a saturated Dieudonné complex and its completion. In this context introduce the category of Dieudonné towers. — [7, §2], [5, §2]

3 Dieudonné algebras (14 June 2023)

In this talk, we want to give the complexes introduced previously some multiplicative structure.

Introduce the category of Dieudonné algebras. Explain how the de Rham-complex is an example and the role of the Cartier isomorphism. Define the category of saturated Dieudonné algebras, discuss their completion and the comparison to Witt vectors. — [7, §2], [5, §3.1-§3.6]

(This can also be joined with the previous talk.)

4 The saturated de Rham–Witt complex (21 June 2023)

We come now to the main definition.

Show that the category of saturated Dieudonné algebras has an initial object, and define the saturated de Rham–Witt complex. Show how to globalise this construction. — [7, §2], [5, §4.1, §5.2-§5.3]

To motivate the definition mention the comparison with crystalline cohomology and/or with the classical de Rham Witt complex. — [7, §3.1, §3.2], [5, §4.2-§4.4]

5 Differences between the two constructions (28 June 2023)

Start by recalling some of the differences in the construction. —[5, §1.2]
If there is interest, we could look at some explicit examples for "nice" singularities.
Look at the classical and the saturated de Rham–Witt complex of a cusp and explain the difference.
Discuss invariance under reduction and/or semi-normalisation as generalisation. — [7, §4], [5, §3,§6]
Or: look at the case of a node. — [7, §4]
You can also mention the results by Ogus' on toroidal singularities. — [8]

6 Further developments (05 July 2023)

To finish, we want to get an overview of the current state of art.
You can discuss applications, such as on the Nygaard filtration — [5, §8], [7, §5.2]
Or further developments of the constructions, such as a logarithmic version — [9]
Or (open) questions, concerning for example a relative version, comparison to rigid cohomology for open and singular varieties, finiteness and coherence... — [7, §6]

References

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