A new approach to the de Rham-Witt complex after Bhatt–Lurie–Mathew

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Abstract

These are the notes for a reading/research/working group seminar in Arithmetic Geometry during the summer term 2023, where we study an approach to the de Rham–Witt complex introduced by Bhargav Bhatt, Jacob Lurie and Akhil Mathew.
The talks are given by different members of the Moritz Kerz's working group.

Contents

In arithmetic and algebraic geometry cohomology theories provide important tools. One of the more well-known theories for (smooth) algebraic varieties X over the complex numbers is (algebraic) de Rham cohomology $\Omega^\bullet_{X/\mathbb{C}}$. However, over fields k of positive characteristic p, algebraic de Rham cohomology is a less satisfactory invariant. This is due in part to the fact that it takes values in the category of k-vector spaces, and therefore consists entirely of p-torsion.

To remedy this, Berthelot and Grothendieck introduced *crystalline cohomology* [2] for smooth projective varieties X over a perfect field k of characteristic $p > 0$ as sheaf cohomology of a certain site. These cohomology groups have coefficients in the Witt vectors $W(k)$ and after inverting p, they provide for smooth projective k-varieties a Weil cohomology with p-adic coefficients. Later Bloch and Illusie [8] gave another description of crystalline cohomology, which is closer in spirit to the definition of algebraic de Rham cohomology: they showed that the crystalline cohomology groups can be computed by a complex of sheaves on the étale site, the de Rham–Witt complex $W\Omega^\bullet_X/k$. However crystalline cohomology, and therefore the hypercohomology of the de Rham–Witt complex, do not work well for open or singular varieties over k [3].

In 2017, Bhatt, Lurie and Mathew introduced a new approach to the construction of the de Rham–Witt complex [4]. They called it "Constructing de Rham–Witt complexes on a budget". And indeed, it is simpler in the sense that it avoids lengthy calculations for standard affine spaces that appear in [8].

Moreover, it is potentially of greater interest, as for singular varieties it provides promising substitutes for crystalline cohomology.

In this seminar, we want to study this new approach to the de Rham–Witt complex, and depending on our time constraints consider further developments and (potential) applications. As a guideline we will use Illusie's report [9] on [4].

1 Context and overview (26 April 2023 – Veronika Ertl)

Abstract

In this talk, we want to get some historical context to motivate our studies.
Give some background on the motivation and properties of crystalline cohomology and the de Rham–Witt complex. Recall the décalage of filtrations observed by Deligne for general complexes and explain how it reappears in the context of p-adic Hodge theory.
Explain how the décalage functor emerges and how this motivated the new construction of the de Rham–Witt complex.
Give an overview of the construction of Bhatt–Lurie–Mathew's approach to the de Rham–Witt complex and explain some of the differences to the original construction. — [9, §1], [4, §1.1-§1.3]

1.1 Some background on the classical de Rham–Witt complex

Geometry is often thought of as the study of solutions of polynomial equations. That sounds simple enough: we start with one or several equations in finitely many variables over $\mathbb{Z}$, and look at their solutions in different fields, such as the complex or real numbers or even in finite fields. If we think of them as geometric objects, we can ask ourselves how do they look, are there singularities etc. But we all know that it is not that simple. Depending on the context, we can try to analyse them from an algebraic or analytic perspective and have certain tools to help us on the way. Among these tools are certain invariants which we call "cohomology theories".

Let me give you an example.

Example 1. If we are interested in let's say an elliptic curve X over $\mathbb{F}_p$ – which in fact we can describe with polynomical equations in the integers. In the language of algebraic geometry, this is a projective smooth variety. An important question is to know how many points does this projective smooth variety have. This is not a vacuous question, it has real world applications for example in cryptography. This number of points is encoded in the so called Zeta-function of X. So we can try to analyse this Zeta function. The famous Weil conjectures concern properties of this Zeta function (such as rationality, a functional equation, the locus of it's zeros and poles), and would follow from the existence of a cohomology theory with certain properties, which would then be called Weil cohomology.

Definition 2. A Weil cohomology is a functor

$$H^* : \left\{ \begin{array}{c} \text{smooth proper } k\text{-varieties} \\ k \text{ an arbitrary field} \end{array} \right\} \to \left\{ \begin{array}{c} \text{graded } K\text{-algebras} \\ K \text{ a field of characteristic } 0 \end{array} \right\}$$

satisfying certain properties, such as Poincaré duality, Künneth formula, or the existence of cycle classes.

Example 3. If $k = \mathbb{C}$, then the (algebraic) de Rham cohomology is a Weil cohomology. In that case, we have $k = K = \mathbb{C}$. The algebraic de Rham cohomology $H^*_{\mathrm{dR}}(X/k)$ of a smooth variety X is the hyper cohomology of the de Rham complex

$$\Omega^0_{X/k} \xrightarrow{d} \Omega^1_{X/k} \xrightarrow{d} \Omega^2_{X/k} \xrightarrow{d} \cdots$$

When $k = \mathbb{C}$, Grothendieck showed [6] that the algebraic de Rham cohomology is isomorphic to the usual de Rham cohomology of the underlying complex manifold, and therefore also to the singular cohomology with complex coefficients of the topological space $X(\mathbb{C})$.

In the case that k is a finite field, the algebraic de Rham cohomology is not such a good invariant. One reason is, that the resulting cohomology groups are k-vector spaces, and therefore consist entirely of p-torsion. And in fact, a Weil cohomology should have cohomology groups with coefficients in characteristic 0. So first we have to think about what the coefficients for our cohomology theory should be.

If we think about the complex case, the complex numbers can be obtained from the rational numbers by first completing with respect to the usual archimedean norm $|\cdot|_\infty$ and then taking the algebraic closure

$$\mathbb{Q} \hookrightarrow \widehat{\mathbb{Q}} = \mathbb{R} \hookrightarrow \overline{\mathbb{R}} = \mathbb{C}$$

But we can do the same thing for any non-archimédian norm $|\cdot|_\ell$, where ℓ is any prime number and $|x|_\ell = \left(\frac{1}{x}\right)^{\operatorname{ord}_\ell(x)}$for $x \in \mathbb{Q}$, and obtain

$$\mathbb{Q} \hookrightarrow \widehat{\mathbb{Q}} = \mathbb{Q}_\ell \hookrightarrow \overline{\mathbb{Q}}_\ell \hookrightarrow \widehat{\overline{\mathbb{Q}}}_\ell = \mathbb{C}_\ell$$

This brings us to ℓ-adic coefficients and their integer versions:

$$\mathbb{Z}_\ell := \{x \in \mathbb{Q}_\ell \mid |x|_\ell \leqslant 1\}$$

It turns out that if we take $\ell = p$ this is a very interesting object, as it incorporates th information of $\mathbb{F}_p$ which is its residue field $\mathbb{Z}_p/p\mathbb{Z}_p = \mathbb{F}_p$.

There is a canonical analogue for any perfect field k of characteristic $p > 0$, the ring of Witt vectors $W(k)$ [12].

Definition 4. For a perfect field k of characteristic $p > 0$, a ring W is a p-ring with residue field k, if there is $\pi \in W$ such that W is separated for the π-adic topology and complete, and $k = W/\pi$. The ring W is said to be strict if $p = \pi$.

Theorem 5. *Up to canonical isomorphism, there is a unique strict p-ring $W(k)$ with residue field k such that there is a unique multiplicative set of representatives $\tau : k \to W(k)$, called Teichmüller representatives.*

Definition 6. For a perfect field k of characteristic $p > 0$, the ring of Witt vectors $W(k)$ is the unique (up to canonical isomorphism) strict p-ring (meaning separated and complete for the p-adic topology) with residue field $k = W(k)/pW(k)$ and a unique multiplicative set of representatives $\tau : k \to W(k)$ called Teichmüller representatives.

Geometrically, $\operatorname{Spec} W(k)$ has two points, a closed point and a generic point

$$\operatorname{Spec}(k) \hookrightarrow \operatorname{Spec}(W(k)) \hookleftarrow \operatorname{Spec}(K)$$

where $K = \operatorname{Frac}(W(k))$. It also has some nice features, such as extra structure in the form of operators F, V called Frobenius and Verschiebung, satisfying $FV = p = VF$. The proof of the existence of $W(k)$ is constructive, so that one obtains a very explicit description in terms of "power series in p".

The underlying set of the ring $W(k)$ is given by $k^{\mathbb{N}}$, with addition and multiplication given by certain universal polynomials. There is a map

$$w : W(A) \to A^{\mathbb{N}}$$

given in each coordinate by polynomials determined recursively such that the map becomes a ring homomorphism. The Teichmüller map is given by

$$\tau : k \to W(k), a \mapsto (a, 0, \ldots) =: [a],$$
$$\tau : k \to k^{\mathbb{N}}, a \mapsto (a, a^p, a^{p^2}, \ldots),$$

the Verschiebung by

$$V : W(k) \to W(k), (a_0, a_1, \ldots) \mapsto (0, a_0, a_1, \ldots),$$
$$V : k^{\mathbb{N}} \to k^{\mathbb{N}}, (a_0, a_1, \ldots) \mapsto (0, pa_0, pa_1, \ldots),$$

and the Frobenius by

$$F : W(k) \to W(k), (a_0, a_1, \ldots) \mapsto (a_0^p, a_1^p, \ldots),$$
$$F : k^{\mathbb{N}} \to k^{\mathbb{N}}, (a_0, a_1, \ldots) \mapsto (a_1, a_2, \ldots).$$

The idea prevailing in the middle of the last century was now that if we have a reasonably nice variety X over a perfect field k of positive characteristic $p > 0$, we should lift it to a scheme over $W(k)$ so we have a similar picture as above

$$\begin{array}{ccccc} X & \hookrightarrow & \mathcal{X} & \hookleftarrow & X_K \\ \downarrow & & \downarrow & & \downarrow \\ \operatorname{Spec}(k) & \hookrightarrow & \operatorname{Spec}(W(k)) & \hookleftarrow & \operatorname{Spec}(K) \end{array}$$

And then consider the algebraic de Rham cohomology of the lift. The problem is, that one cannot in general expect a global lift. On the other hand, it is possible to consider local lifts to $W(k)$.

The crystalline cohomology of Berthelot and Grothendieck [2] makes use of the existence of local lifts. With some extra work these data can be made into a site. The miracle now is, that if one takes the cohomology of the structure sheaf of this site, one obtains in fact a Weil cohomology theory. But it is not so satisfactory, as it is a rather abstract definition.

Here is, where the de Rham–Witt complex due to Bloch–Deligne–Illusie [8] comes in as an alternative way to compute crystalline cohomology closer in spirit to algebraic de Rham cohomology. Roughly, the idea is not to lift the k-variety X, but to lift the de Rham complex $\Omega^\bullet_{X/k}$ as a sheaf to $W(k)$, and promote the operators F, V to operators on this complex. Just as the ring of Witt vectors, it can be defined through a universal property.

Definition 7. Let X be a k-variety and $X_{\text{ét}}$ its étale topos. A de Rham-V-procomplex on X is a projective system

$$M_\bullet = ((M_n)_{n\in\mathbb{Z}}, R : M_{n+1} \to M_n)$$

of $\mathbb{Z}$-dga's on $X_{\text{ét}}$ and a family of additive maps

$$(V : M_n^i \to M_{n+1}^i)_{n\in\mathbb{Z}}$$

such that $RV = VR$ satisfying the following conditions:

(V1) $M_{n\leqslant 0} = 0$, M_1^0 is an $\mathbb{F}_p$-algebra and $M_n^0 = W_n(M_1^0)$ where R and V are the usual maps.

(V2) For $x \in M_n^i$ and $y \in M_n^j$
$$V(xdy) = (Vx)dVy.$$

(V3) For $x \in M_1^0$ and $y \in M_n^0$
$$(Vy)d[x] = V([x]^{p-1}y)dV[x].$$

A morphism of de Rham-V-procomplexes is a morphism of a projective system of dga's $(f_n : M_n \to M'_n)_n$ compatible with all the additional structure in the obvious way ($f_{n+1}V = Vf_n$ and $f_n^0 = W_n(f_1^0)$). Thus the de Rham-V-procomplexes form in a natural way a category denoted by $\mathrm{VDR}(X)$.

Theorem 8. *The category of de Rham-V-procomplexes* $\mathrm{VDR}(X)$ *has an initial object, called the de Rham–Witt complex. More precisely, the forgetful functor*

$$\mathrm{VDR}(X) \to \mathbb{F}_p - \mathrm{Alg}(X) \quad , \quad M_\bullet \mapsto M_1^0$$

has a left adjoint $W_\bullet\Omega^\bullet_{X/k}$. We write $W\Omega^\bullet_{X/k}$ for the projective limit.

Definition 9. For a variety X/k, the de Rham–Witt complex $W\Omega^\bullet_{X/k}$ is a complex of étale sheaves which is the projective limit of the initial object in a category called de Rham-V-procomplexes. We write $W\Omega^\bullet_{X/k}$ for the projective limit.

Again the proof for the existence of the de Rham–Witt complex is constructive, so that there is a rather explicit description. Out of this construction also comes a Frobenius operator F, satisfying $FdV = d$ and $FV = p = VF$, so there is no need to include it into the definition. As we will see, the new construction that we will discuss in this seminar does it in some sense the other way round.

Moreover, the construction also provides a canonical morphism of complexes of étale sheaves $W\Omega^\bullet_{X/k} \to \Omega^\bullet_{X/k}$ which induces a quasi-isomorphism

$$W\Omega^\bullet_{X/k}/pW\Omega^\bullet_{X/k} \xrightarrow{\sim} \Omega^\bullet_{X/k}.$$

This emphasises the close link to the de Rham complex. But there is another link:

Theorem 10. *Assume that X lifts to a smooth formal scheme $\mathcal{X}/W(k)$ together with a lift of Frobenius $\varphi_\mathcal{X}$, then there is a natural quasi-isomorphism*

$$\Omega^\bullet_{\mathcal{X}/W(k)} \xrightarrow{\sim} W\Omega^\bullet_{X/k}$$

of complexes of abelian sheaves on the underlying topological space of X.

Remark 11. Note that while the above map of complexes depends on the choice of a Frobenius lift $\varphi_\mathcal{X}$, the induced map on cohomology is independent thereof.

1.2 A new approach to the de Rham–Witt complex

The proofs of the above results depend on some rather laborious calculations. The goal of the construction in [4] was to obtain an alternative construction

- which agrees with the classical de Rham–Witt complex in the case of smooth varieties,
- but can be used to show the above results in a calculation-free way.

The essential ingredients are the so-called Cartier isomorphism and some basic homological algebra. As mentioned above, the resulting complex coincides with the classical one for smooth varieties. However, they differ in general. We follow the convention in [4] to distinguish them we denote and call them as follows:

$W\Omega^\bullet_{X/k}$	the classical de Rham–Witt complex
$\mathcal{W}\Omega^\bullet_{X/k}$	the saturated de Rham–Witt complex

Let us come to an overview of the construction which will be the main focus of the next couple talks. We discuss the affine case, that is we consider the spectrum $X = \operatorname{Spec}(R)$ of a smooth k-algebra.

Naively, we first consider choose a lift of R to $W(k)$, that is, we take a p-adically complete p-torsin free $W(k)$-algebra $\widetilde{R}$ together with an isomorphism $R \cong \widetilde{R}/p\widetilde{R}$. Let $\widehat{\Omega}^\bullet_{\widetilde{R}/W(k)}$ denote the p-adic completion of the de Rham complex of $\widetilde{R}$ over $W(k)$. This is a commutative differential graded algebra over $W(k)$ and there is a canonical isomorphism $\Omega^\bullet_{R/k} \cong \widehat{\Omega}^\bullet_{\widetilde{R}/W(k)}/p\widehat{\Omega}^\bullet_{\widetilde{R}/W(k)}$. However, it depends on the choice of $\widehat{R}$ and not functorially on R. Next we also consider a lift of Frobenius $\varphi : \widetilde{R} \to \widetilde{R}$ (meaning we have $\varphi(x) = x^p$ mod p. Then φ is divisible by p^n on $\widehat{\Omega}^n_{\widetilde{R}/W(k)}$, so that we obtain a morphism of graded algebras

$$\begin{aligned} F :& \widehat{\Omega}^\bullet_{\widetilde{R}/W(k)} \to \widehat{\Omega}^\bullet_{\widetilde{R}/W(k)}, \\ & F(a_0 da_1 \wedge \cdots da_n) = \varphi(a_0)\frac{d\varphi(a_1)}{p}\cdots\frac{d\varphi(a_1)}{p}. \end{aligned}$$

Note that F is not a map of differential graded algebras, as it does not commute with d, but instead $dF(\omega) = pF(d\omega)$.

This motivates the definition of the category of Dieudonné complexes, which provide a basic framework for the construction and is the content of the next talk.

Definition 12. A Dieudonné complex is a triple $(M^\bullet, d, F)$, where $(M^\bullet, d)$ is a (cochain) complex of abelian groups and $F : M^\bullet \to M^\bullet$ a map of graded abelian groups called Frobenius satisfying $dF = pFd$.

For any Dieudonné complex $(M^\bullet, d, F)$, the Frobenius map F induces a map of complexes

$$(M^\bullet/pM^\bullet, 0) \to (M^\bullet/pM^\bullet, d) \tag{1.1}$$

In the case of the Diedonné complex $(\widehat{\Omega}^\bullet_{\widetilde{R}/W(k)}, d, F)$ this map is a quasi-isomorphism (more precisely, on cohomology groups, it induces a Frobenius-semilinear isomorphism $\Omega^\bullet_{R/k} \xrightarrow{\sim} H^\bullet_{\mathrm{dR}}(\mathrm{Spec}\,(R)/k)$). This is the inverse of the classical Cartier isomorphism.

This motivates the following notion:

Definition 13. A Dieudonné complex $(M^\bullet, d, F)$ is said to be of Cartier type if it is p-torsion-free and (1.1) is a quasi-isomorphism.

Now a crucial point is the construction on the category of Dieudonné complexes $\mathcal{W}\mathrm{Sat}$ called completed saturation. It consists of a saturation process, which produces a new Dieudonné complex equipped with a Verschiebung V, and then a completion with respect to V. This is particularly nice for Dieudonné modules of Cartier type.

Theorem 14. *If $(M^\bullet, d, F)$ is a Dieudonné complex of Cartier type, the canonical map $M^\bullet \to \mathcal{W}\mathrm{Sat}\,(M^\bullet)$ becomes a quasi-isomorphism after reducing modulo p.*

We will see later that saturated de Rham–Witt complex $\mathcal{W}\Omega^\bullet_{R/k}$ can be identified with the completed saturation $\mathcal{W}\mathrm{Sat}\,(\widehat{\Omega}^\bullet_{\widetilde{R}/W(k)})$ of $\widehat{\Omega}^\bullet_{\widetilde{R}/W(k)}$. Moreover, it turns out that it is endowed with an algebra structure that is compatible with its Dieudonné complex structure.

This brings us to the topic of the third talk: Dieudonné algebras.

Definition 15. A strict Dieudonné algebra is a Dieudonné complex $(A^\bullet, d, F)$ with the structure of a differential graded algebra such that:

(i) $A^i = 0$ for $i < 0$.

(ii) $A^\bullet$ is p-torsion free.

(iii) $F : A^\bullet \to A^\bullet$ is a morphism of graded rings.

(iv) For every $i \in \mathbb{Z}$, the map $F : A^i \to A^i$ is injective and its image consists of those elements x, such that dx is divisible by p.

(v) For $x \in A^0$, $F(x) \equiv x^p \mod p$.

(vi) $A^\bullet$ satisfies a certain completeness condition with respect to the Verschiebung V.

The Frobenius map on R induces a ring homomorphism

$$R \to H^0(\mathcal{W}\Omega^\bullet_{R/k}/p\mathcal{W}\Omega^\bullet_{R/k}),$$

(which is analogous to observing that the de Rham differential on $\Omega^\bullet_{R/k}$is linear over the subring of p-th powers in R). In the context of strict Dieudonné algebras this map can be used to characterise $\mathcal{W}\Omega^\bullet_{R/k}$ via a universal property, much like the classical de Rham–Witt complex, or even the de Rham complex. This is the content of the fourth talk.

Theorem 16. *Let $(A^\bullet, d, F)$ be a strict Dieudonné algebra. Then every ring homomorphism $R \to H^0(A^\bullet/pA^\bullet)$ extends uniquely to a map of strict Dieudonné algebras $\mathcal{W}\Omega^\bullet_{R/k} \to A^\bullet$.*

It turns out that this construction compares with the classical de Rham–Witt complex in the smooth case, but differs in general. This is the content of the fifth talk, where we will look at some examples of singularities. For instance, the saturated de Rham–Witt complex $\mathcal{W}\Omega^\bullet_{R/k}$ does not depend on R, but only on its seminormalisation, which is not true for the classical de Rham–Witt complex $W\Omega^\bullet_{R/k}$. One nice feature is, that $H^0(\mathcal{W}\Omega^\bullet_{R/k})$ can be identified with the seminormalisation R^{sn}. The question whether there is a similarly nice description of the higher cohomology groups seems to be open.

Already the basic definition of the saturated de Rham–Witt complex has some nice applications, such as concerning the Nygaard filtration, whiwe want to touch on in the last talk. One can easily imagine further developments, such as a logarithmic version which was worked out by Yao [1], but also a relative version (for the classical de Rham–Witt complex this was done by Langer–Zink [10]) or a big version (for the classical case done by Hesselholt–Madsen [7]).

1.3 Motivation via the décalage functor

While we will probably not touch on the décalage functor during the seminar, it is quite interesting, as it provided the original motivation for the authors to develop the saturated de Rham–Witt complex.

Consider a smooth $\mathbb{F}_p$-algebra R and its algebraic de Rham complex $\Omega^\bullet_{R/\mathbb{F}_p}$, which as a complex of $\mathbb{F}_p$-vector spaces can be regarded as an element n the derived category $D(\mathbb{F}_p)$. Berthelot–Grhothendiecks crystalline cohomology provides an object $R\Gamma_{\text{cris}}(R/\mathbb{Z}_p)$ in the derived category $D(\mathbb{Z})$ which plays the role of a lift of $\Omega^\bullet_{R/\mathbb{F}_p}$ to characteristic 0. It can be realised by an injective resolution of the structure sheaf of the crystalline site. Thus a priori, if we want to regard it as an explicit complex, it depends on the choice of an injective resolution. Thus it is somewhat surprising, that there is a canonical representative in the form of the de Rham–Witt complex. The construction of Bhatt–Lurie–Mathew provide an explanation for this phenomenon.

Namely, the derived category $D(\mathbb{Z})$ has an additive endo-functor

$$L\eta_p : D(\mathbb{Z}) \to D(\mathbb{Z}).$$

The functor η_p is called décalage functor, as it can be defined using a phenomenon observed by Deligne, when he compared the spectral sequences associated to the canonical and the naïve filtrations of the de Rham complex. It reappeared in the context of p-adic Hodge theory, as Deligne suggested it as a technique to generalise Mazur's theorem of Katz's conjecture concerning the relation of the Hodge and the Newton polygon of crystalline cohomology. Following this suggestion, Ogus proved, that it could be obtained from the following result:

Theorem 17. *The Frobenius endomorphism of R induces a canonical isomorphism in $D(\mathbb{Z})$*

$$\widetilde{\varphi} : R\Gamma_{\text{cris}}(R/\mathbb{Z}_p) \cong L\eta_p R\Gamma_{\text{cris}}(R/\mathbb{Z}_p).$$

This means, that one can regard the pair $(R\Gamma_{\text{cris}}(R/\mathbb{Z}_p), \widetilde{\varphi})$ as a fixed point of $L\eta_p$. Moreover, Bhatt–Morrow–Scholze [5] observed, that for any $K \in D(\mathbb{Z})$, the object $(\mathbb{Z}/p^n\mathbb{Z})\widehat{\otimes}^L_{\mathbb{Z}}(L\eta_p)^n K \in D(\mathbb{Z}/p^n\mathbb{Z})$ admits a canonical representative given by the Bockstein complex. Consequently, if K is equipped with a quasi-isomorphism $\alpha K \cong L\eta_p K$, then

$$(\mathbb{Z}/p^n\mathbb{Z})\widehat{\otimes}^L_{\mathbb{Z}} K \cong (\mathbb{Z}/p^n\mathbb{Z})\widehat{\otimes}^L_{\mathbb{Z}}(L\eta_p)^n K$$

has also a canonical representative, and one might hope that they can be amalgamated to a representative of K itself.

This heuristic can be made precise in the context of Dieudonné complexes:

Theorem 18. *The category of strict Dieudonne complexes DC_{str} is equivalent to the category $\widehat{D(\mathbb{Z})}_p^{L\eta_p}$ of fixed points for the endofunctor $L\eta_p$ on the p-completed derived category of abelian groups.*

Consequently, Ogus' quasi-isomorphism from Theorem 17 guarantees that $R\Gamma_{\text{cris}}(R/\mathbb{Z}_p)$ admits a canonical presentation by a strict Dieudonne complex, namely the de Rham–Witt complex.

2 Dieudonné complexes (7 June 2023)

Abstract

In this talk, we start with basic definitions.
Define the category of saturated Dieudonné complexes and (saturated) Dieudonné complexes and discuss the saturation functor. Give an example.
Define the completion of a saturated Dieudonné complex and show its universal property. Discuss the relation between a saturated Dieudonné complex and its completion. In this context introduce the category of Dieudonné towers. — [9, §2], [4, §2]

3 Dieudonné algebras (14 June 2023)

Abstract

In this talk, we want to give the complexes introduced previously some multiplicative structure.
Introduce the category of Dieudonné algebras. Explain how the de Rham-complex is an example and the role of the Cartier isomorphism. Define the category of saturated Dieudonné algebras, discuss their completion and the comparison to Witt vectors. — [9, §2], [4, §3.1-§3.6]

4 The saturated de Rham–Witt complex (21 June 2023)

Abstract

We come now to the main definition.
Show that the category of saturated Dieudonné algebras has an initial object, and define the saturated de Rham–Witt complex. Show how to globalise this construction. — [9, §2], [4, §4.1, §5.2-§5.3]
To motivate the definition mention the comparison with crystalline cohomology and/or with the classical de Rham Witt complex. — [9, §3.1, §3.2], [4, §4.2-§4.4]

5 Differences between the two constructions (28 June 2023)

Abstract

Start by recalling some of the differences in the construction. —[4, §1.2]
If there is interest, we could look at some explicit examples for “nice” singularities.
Look at the classical and the saturated de Rham–Witt complex of a cusp and explain the difference.
Discuss invariance under reduction and/or semi-normalisation as generalisation. — [9, §4], [4, §3,§6]
Or: look at the case of a node. — [9, §4]
You can also mention the results by Ogus’ on toroidal singularities. — [11]

6 Further developments (05 July 2023)

Abstract

To finish, we want to get an overview of the current state of art.
You can discuss applications, such as on the Nygaard filtration — [4, §8], [9, §5.2]
Or further developments of the constructions, such as a logarithmic version — [1]
Or (open) questions, concerning for example a relative version, comparison to rigid cohomology for open and singular varieties, finiteness and coherence... — [9, §6]

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