1 Crystalline cohomology

Let k be a perfect field of characteristic $p > 0$, $W = W(k)$ its ring of Witt vectors, X/k a k-scheme.

We define the crystalline site:

- Objects: commutative diagrams

$$\begin{array}{ccc} U & \overset{i}{\hookrightarrow} & V \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} W_n \end{array}$$

  where $U \subset X$ is a Zariski open, $i : U \hookrightarrow V$ a PD-thickening with PD-structure δ. We denote this by (U, V, δ).

- Morphisms $(U, V, \delta) \to (U', V', \delta')$ such that $U \hookrightarrow U'$ is an open immersion,$V \to V'$ a morphism compatible with devided powers.

- Coverings: families of morphisms $(U_i, V_i, \delta_i) \to (U, V, \delta)$ such that $V_i \to V$ is an open immersion and the V_i cover V.

Sheaves on the crystalline site: $\mathcal{F}$ is given for each (U, V, δ) by a Zariski sheaf on V such that for $(U, V) \to (U', V') \to (U'', V'')$ the induced morphism on sheaves is transitive and for $(U, V) \to (U', V')$ if $V \to V'$ is an open immersion and $U = U' \times V$ the induced morphism is an isomorphism.

The structure sheaf $\mathcal{O}_{X/W_n}$ is given by

$$(U, V, \delta) \mapsto \mathcal{O}_V \, .$$

Now we can consider the crystalline topos, $(X/W_n)_{\text{cris}}$. The point of passing to the crystalline topos is that it is functorial in X as opposed to the crystalline site. That is, a morphism of k-schemes $f : X \to Y$ induces a morphism of topoi

$$f^{-1} : (Y/W_n)_{\text{cris}} \to (X/W_n)_{\text{cris}}.$$

Crystalline cohomology is by definition the cohomology ofthe structure sheaf:

$$\mathrm{H}^i_{\text{cris}}(X/W_n) = \mathrm{H}^i\left((X/W_n)_{\text{cris}}, \mathcal{O}_{X/W_n}\right)$$

and

$$\mathrm{H}^i_{\text{cris}}(X/W) = \varprojlim \mathrm{H}^i_{\text{cris}}(X/W_n).$$

Annotatio 1.1. This is an integral cohomology theory, with coefficients in W_n, respectively W. There is a crystalline Poincaré lemma. If $F : X \to X$ is the absolut Frobenius on X and $\sigma : W_n \to W_n$ the Frobenius induced by the Frobenius on k, F induces by funtoriality a σ-linear morphism on the crystalline cohomology. Assume X/k is proper and smooth purely of dimension d. Then over $\mathrm{Frac}(W)$ crystalline cohomology defines a Weil cohomology.

2 The deRham-Witt complex and comparison

We want to compare crystalline cohomology to other p-adic cohomology theories, like Hodge, étale, Serre. Thus we construct a convenient complex of sheaves, which in addition makes it easier to calculate. “Witt” because of Frobenius, “deRham” because of analogies to deRham cohomology.

The deRham-Witt procomplex as defined by Illusie: projective system $(W_n\Omega^\bullet_X)_{n\in\mathbb{N}_0}$ compatible with Frobenius, Verschiebung and boundary operator. Inductively defined. Satisfying appropriate relations. Taking limits we get the deRham-Witt complex. If X/k is smooth it is p-torsion free. If it is of finite type we have a unique Cartier operator, which is an isomorphism in the smooth case.

If X/k is smooth consider the hypercohomology of the deRham-Witt complex

$$\mathrm{H}^\bullet_{\text{dRW}}(X/W_n) = \mathbb{H}^\bullet(X, W_n\Omega^\bullet_X).$$

Illusie proofs the following theorem:

Theorema 2.1. *If X is a smooth k-scheme, there is a canonical isomorphism*

$$\mathrm{H}^\bullet_{cris}(X/W_n) \to \mathrm{H}^\bullet_{dRW}(X/W_n).$$

More generally, there is an isomorphism in the derived category $D(X, W_n)$ of sheaves of W_n-modules over X

$$Ru_*\,\mathcal{O}_{X/W_n} \xrightarrow{\cong} W_n\Omega^\bullet_X$$

which is functorial in X/k.

3 Rigid cohomology

A first step towards rigid cohomology was Monsky-Washnitzer cohomology. It can be thought of as analogue to algebraic deRham cohomology for smooth affine varieties. I will come back later to the construction of this rational cohomology theory. Berthelot realised that this can be generalised to arbitrary varieties and introduced his rigid cohomology.

Let X be a k-variety openly imersied in a proper k-variety $j : X \to Y$ (exists by Nagata) and a closed immersion $Y \hookrightarrow P_k$, where P is an $\mathcal{O}_K$-scheme, P_K the rigid (or Raynaud) generic fiber (for example ($P = \mathbb{P}^n$ for some n). In particular there is a specialisation map

$$\mathrm{sp} : P_K \to P_k.$$

For a subset $U \in P_k$ we denote $]U[= \mathrm{sp}^{-1}(U)$ the tube of U in P_K. A strict neighborhood of $]X[$ in $]Y[$ is an admissible open subset of $]Y[$ which together with $]Y\backslash X[$ forms an admissible covering of $]Y[$. The rigid cohomology $H^i_{\mathrm{rig}}(X/K)$ of X is then constructed as the direct limit of the deRham cohomologies of strict neighborhoods of $]X[$ in $]Y[$.

An example why we have to look at these neighbourhoods and can't take just the algebraic deRham cohomology of $]X[$.

Exemplum 3.1. The closed unit disc:

We get an exact functor on abelian sheaves on $]Y[$ by

$$j^\dagger\mathcal{E} = \lim_{\to} j_{V*}j_V^{-1}\mathcal{E},$$

takne over strict neighbourhoods V of $]X[$ in $]Y[$.

Now we define

$$\mathrm{H}^i_{\mathrm{rig}}(X/K) = \mathbb{H}^i(]Y[_P, R\,\mathrm{sp}_*\,j^\dagger\Omega^\bullet_{]Y[}),$$

and check that this is independent of choices.

This gives a rational cohomology theory where we have comparison isomorphisms

- with crystallline cohomology if X is smooth and proper.
- with Monsky-Washnitzer cohomology if X is smooth and affine.

It would be nice if we had a description of rigid cohomology by a suitable deRham-Witt complex for general varieteis X this is done by Davis, Langer and Zink.

Let us now talk about the crystalline-rigid comparison theorem in order to understand better, why crystalline cohomology, although defined for arbitrary varieties, is not a good integral model for rigid cohomology if X is not proper.

4 Comparison theorem

We keep pur notation: k a perfect field of characteristic p, R a complete discrete valuation ring of mixed characteristic, of residue field k and field of fractions K, $W = W(k)$, $S = \operatorname{Spec} R$, $\hat{S} = \operatorname{Spf} R$.

Let X be a finite type R-scheme. Associated spaces:

- The generic fiber X_K.
- The closed fiber X_0 over k.
- The formal completion $\hat{X}$ over $\hat{S}$ of X along the closed fiber.
- The K-rigid analytic spaces X_{rig} and X_K^{an}.

There are of course natural maps of sites between these spaces that allow us to compare the different cohomology theories native to these sites. There is for example rigid and formal GAGA and the deRham comparison. But we want to focus on the crystalline-rigis comparison.

Recall that crystalline cohomology is highly sensitive to ramification, due to the need of having devided powers. There was no problem, when we defined it over $W(k)$, but as we consider it now more generally over R, we assume $e \leq p-1$ where e is the absolute ramification index of R. Also let π be a uniformiser of R.

Propositio 4.1. *Let X be a smooth proper k-scheme, assume X admits a closed embedding $i : X \to \mathcal{P}$ into a p-adic formal S-scheme that is smooth at the points of X and that $e \leq p-1$. Then there is a canonical isomorphism*

$$\mathrm{H}_{rig}(X/K) \to \mathrm{H}_{cris}(X/R) \otimes_R K.$$

This is natural in (k-morphisms of) X and compatible with any local finite flat base change $R \to R'$ with R' a discrete valuation ring having absolute ramification index $e \leq p-1$.

Probatio: The first part was proven by Berhtelot, the part about base change can be seen readily once the isomorphism (or rather quasi-isomorphism between appropriate complexes) is established.

As X is proper and properness is a local property, we can assume $X = \overline{X}$, the closure of X in $\mathcal{P}$. Thus the j of the definition of rigid cohomology is the identity and so is $j^\dagger$ and we have

$$\mathrm{H}^i_{\text{rig}}(X/K) = \mathbb{H}^i(]X[_P, R\,\mathrm{sp}_*\,\Omega^\bullet_{]X[}).$$

In this context, the natural map in the derived category of complexes of sheaves of K-modules on the Zariski site of $\mathcal{P}$

$$\mathrm{sp}_*\,\Omega^n_{]X[} \to R\,\mathrm{sp}_*\,\Omega^\bullet_{]X[}$$

is an isomorphism for all n (using that the rigid space of a tubular neighbourhood of an open affine is "quasi-Stein" and Kiehl's Theorem B). Therefore

$$\mathrm{H}^i_{\text{rig}}(X/K) = \mathbb{H}^i(]X[_P, \mathrm{sp}_*\,j^\dagger\Omega^\bullet_{]X[}).$$

On the other hand, let $D_X(\mathcal{P})$ be the divided power envelope of $\mathcal{I}$ and as before set $\hat{D}_X(\mathcal{P}) = \lim D_X(\mathcal{P}_n)$ where $\mathcal{P}_n = \mathcal{P} \otimes_R (R/\pi^n R)$. There is a canonical isomorphism

$$Ri_*RuX/\hat{S}\,\mathcal{O}_{X/\hat{S}} \to \hat{D}_X(\mathcal{P}) \otimes \hat{\Omega}^\bullet_{\mathcal{P}/S}$$

, functorial in the pair $(X, \mathcal{P})$. Since X is separated and quasi-compact, hypercohomology commutes with tensoring by K (Čech theory), so we conclude that the natural map

$$\mathrm{H}^\bullet_{\text{cris}}(X/R) \otimes_R K \to \mathbb{H}^\bullet(X, \hat{D}_X(\mathcal{P}) \otimes \hat{\Omega}^\bullet_{\mathcal{P}/S} \otimes K)$$

is an isomorphism. Thus, to define a rational comparison isomorphism between crystalline and rigid cohomology, it is enough to define a K-linear quasi-isomorphism of complexes

$$\mathrm{sp}_*\,\Omega^\bullet_{]X[} \to \hat{D}_X(\mathcal{P}) \otimes \hat{\Omega}^\bullet_{\mathcal{P}/S} \otimes K.$$

This is done by showing that there is an integrable connection on $\mathrm{sp}_* \mathcal{O}_{]X[}$ and then defining an $\mathcal{O}_{\mathcal{P}}$-linear morphism

$$\mathrm{sp}_* \mathcal{O}_{]X[} \to \hat{D}_X(\mathcal{P}) \otimes K.$$

Compatibility with flat base chnges follows from the fact that the defined morphisms are natural. □

Corollarium 4.2. *With the same notation as above, suppose that the Frobenius endomorphism of k lifts to R as σ, so the K-vector spaces $\mathrm{H}^n_{rig}(X/K)$ and $\mathrm{H}_{cris}(X/R) \otimes K$ are equipped with a canonical σ-linear Frobenius endomorphisms. The comparison isomorphism is then compatible with the Frobenius endomorphisms of both sides.*

PROBATIO: This follows from the functoriality of the defined map. □

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