Our setting is the following: Let $E/\mathbb{Q}_p$ be a finite extension of residue fields $\mathbb{F}_q/\mathbb{F}_p$ where $q = p^{f_E}$. We choose a uniformiser π of $\mathscr{O}_E$ and denote by Frob_q the $f_E{}^{\text{th}}$-power Frobenius morphism of $\mathscr{O}_E$ (or any $\mathscr{O}_E$-algebra of characteristic p). Furthermore, we consider an algebraically closed extension $k/\mathbb{F}_q$, and an algebrically closed extension of k, which is complete with respect to a non-trivial valuation

$$\nu : F \to \mathbb{R} \cup \infty.$$

We assume that ν be trivial on k, and that k identifies with the residue $\mathscr{O}_F/\mathfrak{m}_F$ field of F. (Note: this means, we are in the equi-characteristic case.)

1 $\mathscr{O}_E$-Witt vectors

We star with the classical case. This can be done in even more generality, see for example [6].

1.1 Classical case

Consider the functor

$$\begin{array}{rcl} \mathscr{F} : \mathscr{O}_E\text{-algebras} & \to & \text{Sets} \\ A & \mapsto & A^{\mathbb{N}}. \end{array}$$

An element of $\mathscr{F}(A)$ is denoted by $[x_i]_{i\geq 0}$, with $x_i \in A$. This functor factors on many ways depending on what conditions we impose. One of them is the unique one that defines the Witt vectors.

Definition 1.1. For $n \in \mathbb{N}$ define the n^{th} Witt polynomila with respect to (the divisor-stable set defined by) π to be

$$\mathscr{W}_{n,\pi} = \sum_{i=0}^{n} \pi^i X_i^{q^{n-i}} \in \mathscr{O}_E[X_0, \ldots, X_n].$$

Lemma 1.2. *There is a unique factorisation*

$$\begin{array}{ccc} \mathscr{O}_E\text{-}\mathit{algebras} & \xrightarrow{\quad\mathscr{F}\quad} & \mathit{Sets} \\ & \searrow^{W_{\mathscr{O}_E,\pi}} \qquad \nearrow & \\ & \mathscr{O}_E\text{-}\mathit{algebras} & \end{array}$$

such that the natural transformation, evaluation of the Witt polynomials on A

$$\begin{array}{rcl} \mathscr{W}_{\pi,A} : W_{\mathscr{O}_E,\pi}(A) & \to & A^{\mathbb{N}} \\ [a_i] & \to & (\mathscr{W}_{n,\pi}(a_0, \ldots, a_n)) \end{array}$$

is a morphism of $\mathscr{O}_E$-algebras.

This lemma gives a ring structure to the set $A^{\mathbb{N}}$. The elements $\mathscr{W}_{n,\pi}(a_0, \ldots, a_n)$ are called ghost components. The lemma is an abstract fact, which may be proved without yielding a useful construction, but unicity and functoriality enable us to construct the set algebraically. We will list some of the important properties later.

Proof: It is easy to see, that $W_{\mathscr{O}_E,\pi}(A) = A^{\mathbb{N}}$ as sets.

The next step would be to show that the maps $\mathscr{W}_{n,\pi} : W_{\mathscr{O}_E,\pi}(A) \to A$ are homomorphisms of rings for all $n \in \mathbb{N}$. The ring structure on $W_{\mathscr{O}_E,\pi}(A)$ is given by polynomials over $\mathscr{O}_E[\frac{1}{\pi}$, $S = (S_0, S_1, \ldots)$ for addition, and $P = (P_0, P_1, \ldots)$ for multiplication, where the $S_i, P_j \in \mathscr{O}_E[\frac{1}{\pi}[X_1, Y_1, X_2, Y_2, \ldots]$. These polynomials are defined recursively and a priori with rational coefficients, so that one has to prove they are in fact integers. This is done in general by Zink in [9], but I couldn't get a hold of this book. I believe the main ideas can be found in his lecture notes [8], but there he does it only for p as uniformiser, not π. If I have time, I will check this later.

Once this fact established, we can adapt the classical proof to this more general case. Recall that the codomain $A^{\mathbb{N}}$ of $\mathscr{W}_\pi$ is a ring with zero $(0, 0 \ldots)$, one $(1, 1, \ldots$ and componentwise addition and multiplication. If A is an $\mathscr{O}_E$-algebra without π-torsion (is without p-torsion enough??) and an endomorphism φ lifting the Frobenius, we can solve the the Witt polynomials to get Witt components $x \in W_{\mathscr{O}_E,\pi}(A)$ in terms of the ghost components by cancelling π's. So $\mathscr{W}_\pi$ is in fact injective with image

$$\left\{(x_i) \in A^{\mathbb{N}} \quad \middle| \quad x_{i+1} \equiv \varphi(x_i) \mod \pi^{i+1}\right\},$$

making $W_{\mathscr{O}_E,\pi}(A)$ into a subring of $A^{\mathbb{N}}$. If π is invertible, the map is in fact an isomorphism of rings.

This establishes the existence of the "big Witt vectors" over $\mathscr{O}_E$, and by functoriality this can be extended to any $\mathscr{O}_E$-algebra

Of course here I omitted many details of the proof. □

Some remarks to follow:

- The factorisation is unique up to canonical isomorphism.
- There is a unique system of representatives

$$\begin{array}{rcl} [\cdot] : A & \to & W_{\mathscr{O}_E,\pi}(A) \\ a & \mapsto & [a] = [a, 0, \ldots] \end{array}$$

  compatible with multiplication – in fancy words: adapted to the multiplicative group $\mathbb{G}_m$. This is a formal group, that is a representable functor on the category of profinite R-algebras to sets. The group structure is given by power series $F = (F_i) \in R[[\underline{X}, \underline{Y}]]$ (in this case $F = X + Y + XY$) subject to the axioms

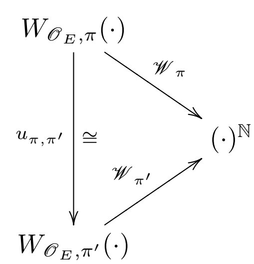
  1. Identity law: $\underline{X} = F(\underline{X}, \underline{0}) = F(\underline{0}, \underline{X})$.
  2. Associativity: $F(\underline{X}, F(\underline{Y}, \underline{Z})) = F(F(\underline{X}, \underline{Y}), \underline{Z})$
  3. Commutativity (if the group is commutative): $F(\underline{X}, \underline{Y}) = F(\underline{Y}, \underline{X})$.

  The inverse law is automatic as we can (recursively) always find a power series $G(X)$ such that $F(X, G(X)) = 0$. From the axioms it also follows, that F is always of the form $X+Y+$terms of higher order. These are the usual Teichmüller representatives.

- If A is a perfect $\mathbb{F}_q$-algebra, every element can be written uniquely in the form $x = \sum_{n=0}^{\infty} [x_n] \pi^n$ with $x_i \in A$. The explicit construction of the polynomials determining addition and multiplication depends on the fact that the Teichmüller representative commutes with multiplication.
- The formation of $W_{\mathscr{O}_E,\pi}(A)$ and [] is functorial in A.
- The image of $\mathscr{W}_\pi$ only depends on the ideal generated by π and not on the choice of π itself. For a different choice of uniformiser there exists a unique isomorphism of functors

$$u_{\pi,\pi'} : W_{\mathscr{O}_E,\pi} \to W_{\mathscr{O}_E,\pi'},$$

  making the following diagram commutative:



  The u's are transitive in π: $u_{\pi',\pi''} \circ u_{\pi,\pi'} = u_{\pi,\pi''}$.

Definition 1.3. Set

$$W_{\mathscr{O}_E} = \lim_{\leftarrow} W_{\mathscr{O}_E,\pi} : \{\mathscr{O}_E\text{-algebras}\} \to \{\mathscr{O}_E\text{-algebras}\}$$

where the limit goes over all the uniformisers in $\mathscr{O}_E$. We consider as well the composition of morphisms

$$\mathscr{W} : W_{\mathscr{O}_E}(A) \xrightarrow{\cong} W_{\mathscr{O}_E,\pi}(A) \xrightarrow{\mathscr{W}_\pi} A^{\mathbb{N}}.$$

This now does not anymore depend on the choice of a uniformiser. The same holds for the Teichmüller representatives as for $a \in A$ it is easy to verify that $u_{\pi,\pi'}([a]_\pi) = [a]_{\pi'}$, so it is justified to write $[\cdot]$ independent of the choice of uniformiser.

- There is a unique endomorphism independent of the choice of π

$$F : W_{\mathscr{O}_E}(\cdot) \to W_{\mathscr{O}_E}(\cdot),$$

such that for $a \in W_{\mathscr{O}_E}(A)$ and $\mathscr{W}(a) = (x_i)$, then $\mathscr{W}(F(a)) = (x_{i+1})$. This is the Frobenius, which is easiest seen looking at the powerseries induced by the Witt polynomials.

- The morphism Verschiebung is dependent of the choice of π

$$V_\pi : W_{\mathscr{O}_E}(\cdot) \xrightarrow{\cong} W_{\mathscr{O}_E,\pi}(\cdot) \xrightarrow{V_\pi} W_{\mathscr{O}_E,\pi}(\cdot) \xrightarrow{\cong} W_{\mathscr{O}_E}(\cdot),$$

where the middle morphism is the shift on $W_{\mathscr{O}_E,\pi}$, $[a_i] \to [0, a_0, a_1, \ldots]$.

- We have the following properties:
  - $FV_\pi = \pi$.
  - $V_{\pi'} = \frac{\pi'}{\pi} v_\pi$.
  - $V_\pi(F(x).y) = x.V_\pi(y)$.
  - From these properties we deduce that $\forall n \in \mathbb{N}, \quad V_\pi^n W_{\mathscr{O}_E}$ is an ideal in $W_{\mathscr{O}_E}$ independent of the choice ofπ.
  - $W_{\mathscr{O}_E}$ is V_π-adically complete. The induced topologies depending on π are of course equivalent. So

$$W_{\mathscr{O}_E}(A) \xrightarrow{\cong} \lim_{\leftarrow} W_{\mathscr{O}_E}(A)/V_\pi^n,$$

    and every element $a \in W_{\mathscr{O}_E}(A)$ can be written uniquely as $\sum V_\pi^n [a_n]$.
  - If A is an $\mathbb{F}_q$-algebra, $V_\pi F = \pi$ and $F(\sum V_\pi^n [a_n]) = \sum V_\pi^n [a_n^q]$.
  - If A is in addition perfect, $W_{\mathscr{O}_E}(A)$ is π-addically complete, π-torsion free, every element can be written uniquely in the form $x = \sum_{n=0}^{\infty} [x_n]\pi^n$ with $x_i \in A$. Moreover, $W_{\mathscr{O}_E}(A)$ is up to isomorphism the unique lift of A which is π-torsion free.

- For a field extension E'/E it is easy to see

Lemma 1.4. *For an $\mathscr{O}_{E'}$-algebra A, there is a unique natural morphism of $\mathscr{O}_E$-algebras*

$$u : W_{\mathscr{O}_E}(A) \to W_{\mathscr{O}_{E'}}(A),$$

such that the diagram

$$\begin{array}{ccc} W_{\mathscr{O}_E}(A) & \xrightarrow{u} & W_{\mathscr{O}_{E'}}(A) \\ & {\scriptstyle (\mathscr{W}_{f_{E'/E},n})_n} \searrow \quad \swarrow {\scriptstyle \mathscr{W}} & \\ & A^{\mathbb{N}} & \end{array}$$

commutes. We have

$$\begin{aligned} u([a]) &= [a]\,, \\ u(V_\pi x) &= \frac{\pi}{\pi'} V_{\pi'}({F^{f_{E'/E}}}^{-1} u(x)), \\ u(F^{f_{E'/E}} x) &= Fu(x). \end{aligned}$$

We have a natural morphism

$$W_{\mathscr{O}_E}(A) \otimes_{\mathscr{O}_{E_0'}} \mathscr{O}_{E'} \to W_{\mathscr{O}_{E'}}(A),$$

where E_0' is the maximal unramified extension of E inside E'. If A is a perfect $\mathbb{F}_{q'}$-algebra ($\mathbb{F}_{q'}$ the residue field of E') this is an isomorphism as in this case $W_{\mathscr{O}_{E'}}(A)$ is the unique π'-adic π'-torsion free lift of A as mentioned above.

Example 1.5. If E_0 is the maximal unramified extension of $\mathbb{Q}_p$ in E, W the usual Witt vectors (belonging to $\mathbb{Q}_p$), then for a perfect $\mathbb{F}_q$-algebra A there is a canonical isomorphism

$$\begin{aligned} W(A) \otimes_{\mathscr{O}_{E_0}} &\xrightarrow{\cong} W_{\mathscr{O}_E}(A) \\ [a] \otimes 1 &\mapsto [a] \\ F^{f_E} \otimes \mathrm{id} &\leftrightarrow F \end{aligned}$$

1.2 Twisted Witt vectors

We now consider some deformations of the Teichmüller representatives. We mentioned earlier that the Teichmüller representatives are addapted to the multiplicative (formal) group $\mathbb{G}_m$. One could ask, if we still get a good theory of Witt vectors if we replce the Teichmüller representatives by a system of representatives that are adapted to a different formal group, where the group law is given by a diffenrent set of power series. As mentioned existence and uniqueness of the Witt vectors can be proven quite formally, however the constructible proof would change. Although this question might be very interesting in a general setting, we will focus on the Lubin-Tate group associated to the field E, as it turns out to be convenient.

Starting from the same set-up as above. The idea of the Lubin-Tate group (or more general module) is that it reflects the dominating principle of class field theory, to the effect that prime elements correspond to Frobenius elements.

Definition 1.6. A Lubin-Tate module over $\mathscr{O}_E$ for the prime element π is a formal $\mathscr{O}_E$-module F such that

$$\pi_F(X) = X^q \mod \pi,$$

where π_F denotes the action of π on F.

If we reduce the coefficients $\mod \pi$ we obtain a formal group $F(X,Y)$ over the residue class field $\mathbb{F}_q$. The reduction $\mod \pi$ of $\pi_F(X)$ is an endomorphism of this group. On the other hand, X^q is the frobenius endomorphism of this group. So, F is a Lubin-Tate module, if the endomorphism defined by the action of π gives via reduction the Frobenius endomorphism.

In other words, the Lubin-Tate formal group law over $\mathscr{O}_E$ is the unique (1-dimensional) formal group law $F(X,Y)$ such that $e(X) = \pi X + X^q$ is an endomorphism of F, i.e.

$$e(F(X,Y)) = F(e(X), e(Y)).$$

More generally one allows any formal power series e such that

$$\begin{aligned} e(X) &= \pi X + \text{higher order terms} \\ e(X) &= X^q \mod \pi \end{aligned}$$

Under these conditions, a formal group law is strictly isomorphic. Moreover, the derivative at the origine is the prime element and reduction modulo the maximal ideal gives the Frobenius. For each element $a \in \mathscr{O}_E$ there is a unique endomorphism f on F such that $f(x) = ax +$ higher order. So $\mathscr{O}_E$ acts on the formal group.

The compatibility of the Teichmüller representatives with multiplication in the usual case, can be explained by the nature of the Witt polynomials. We will replace the powers of variables X_i by polynomials. Let $Q \in \mathscr{O}_E[X]$ be such that $Q \equiv X^q \mod \pi$. Now set

$$\begin{aligned} Q_0 &= X \\ Q_n &= \underbrace{Q \circ \cdots \circ Q}_{n\text{-times}} \qquad \forall n \in \mathbb{N}. \end{aligned}$$

Our altered Witt polynomials now depending on Q are

$$\mathscr{W}_{n,Q,\pi} = \sum_{i=0}^{n} \pi^i Q_{n-i}(X_i) \in \mathscr{O}_E[X_0, \ldots, X_n],$$

(note, if we use $Q = X^q$ we recover the "old" Witt polynomials). From now on, we just have to adapt the theory described in the previous section. As before we have the functor

$$\mathscr{F} : \mathscr{O}_E\text{-algebras} \to \text{Sets}$$

however now, we get a different factorisation of this functor.

Proposition 1.7. *There is a unique factorisation*

$$\begin{array}{ccc} \mathscr{O}_E\text{-}algebras & \xrightarrow{\quad\mathscr{F}\quad} & Sets \\ & {\scriptstyle W_{\mathscr{O}_E,Q,\pi}} \searrow \quad \nearrow & \\ & \mathscr{O}_E\text{-}algebras & \end{array}$$

such that the natural transformation, evaluation of the Q-Witt polynomials on A

$$\begin{aligned} \mathscr{W}_{\pi,Q,A} : W_{\mathscr{O}_E,Q,\pi}(A) &\to A^{\mathbb{N}} \\ [a_i] &\to (\mathscr{W}_{n,Q,\pi}(a_0, \ldots, a_n)) \end{aligned}$$

is a morphism of $\mathscr{O}_E$-algebras.

Proof: This is very similar to the proof before. We have the same ring structure on $A^{\mathbb{N}}$ as before, namely componentwise. Recursively we obtain series of polynomials in $\mathscr{O}_E[\frac{1}{\pi}$ (I'm not sure about this since there are q^{th} roots involved now) $S = (S_0, S_1, \ldots)$ and $P = (P_0, P_1, \ldots)$, which define addition and multiplication on the domain. We have to show, that the coefficients are in $\mathscr{O}_E$. One main ingredient in the classical case is the fact that if $x \equiv y \mod \pi$ then $x^{q^i} \equiv y^{q^i} \mod \pi^{i+1}$. This can easily be generalised for Q:

Lemma 1.8. *For an $\mathscr{O}_E$-algebra A, and $x, y \in A$, $i \in \mathbb{N}$ such that $x \equiv y \mod \pi^i$, we have $Q(x) \equiv Q(y) \mod \pi^{i+1}$.*

Note that this is a little weaker, but serves the purpose as well. Now we do similar calculations as in the classical case. Reducing to the case of the "big" functor again, this leads to the fact, that for a π-torsion free $\mathscr{O}_E$-algebra with a Frobenius lift φ the morphism

$$\mathscr{W}_{\pi,Q,A} : W_{\mathscr{O}_E,Q,\pi}(A) \to A^{\mathbb{N}}$$

is injective with image

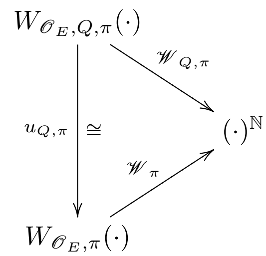
$$\left\{ (x_i) \in A^{\mathbb{N}} \quad \middle| \quad x_{i+1} \equiv \varphi(x_i) \mod \pi^{i+1} \right\}.$$

Now everything works as before. □

As the classical and the twisted constructions are functorial, having good universal properties, there is a unique isomorphism

$$u_{Q,\pi} : W_{\mathscr{O}_E,Q,\pi} \to W_{\mathscr{O}_E,\pi}$$

rendering the diagram



commutative. Composing this isomorphism with the one from the previous section $W_{\mathscr{O}_E,Q,\pi} \xrightarrow{\cong} W_{\mathscr{O}_E,\pi} \xrightarrow{\cong} W_{\mathscr{O}_E}$ we obtain the Q-Teichmüller lift.

Proposition 1.9. *There is a unique natural map with domain the $\mathscr{O}_E$-algebra A*

$$[\cdot]_Q : A \to W_{\mathscr{O}_E}(A)$$

verifying

- $\mathscr{W}([a]_Q) = (Q_n(a))_{n\geqslant 0}$.
- $Q([a]_Q) = [Q(a)]_Q$.
- *Every element of $W_{\mathscr{O}_E}(A)$ can be written uniquely in the form*

$$\sum_{n\geqslant 0} V_\pi^n [a_n]_Q .$$

- *If A is a perfect $\mathbb{F}_q$-algebra, every element of $W_{\mathscr{O}_E}(A)$ can be written uniquely in the form*

$$\sum_{n\geqslant 0} [x_n]_Q \pi^n.$$

  Moreover in this case, the Q-Teichmüller lift is the unique lift under the condition that $Q([x]_Q) = [x^q]_Q$ and we have

$$[x]_Q = \lim_{n\to\infty} Q_n(\left[x^{q^{-n}}\right]).$$

  More generally, if $x \in A$, $\hat{x}_n$ any lift of $x^{q^{-n}}$ then $[x]_Q$ is given by the formula

$$[x]_Q = \lim_{n\to\infty} Q_n(\hat{x}_n).$$

- *If $Q(X) = X^q$, the old and the new Teichmüller lift coincide $[a]_Q = [a]$.*
- *If $E = \mathbb{Q}_p$ and $Q(X) = (1+X)^p - 1$ we have $[a]_Q = [1+a] - 1$*

Remark 1.10. The properties show that formal group laws (such as Lubin-Tate) furnish a generalisation of p^{th} root of unity (and provide an explicit version of the local reciprocity law in the corresponding extension of $\mathbb{Q}_p$, for those who know local field theory). The multiplicative group $\mathbb{G}_m$ is a Lubin-Tate module for the cyclotomic extension with $Q(X) = (1+X)^p - 1$.

For Lubin-Tate we require in addition

$$Q(X) \equiv \pi X \mod X^2.$$

Let $\mathscr{L}\mathscr{T}_Q \in \mathscr{O}_E[\![X,Y]\!]$ be the formal (unique) group law that commutes with Q. Consider a perfect $\mathbb{F}_q$-algebra A and $x, y \in A$ such that A is complete and separated (Hausdorff??) for the (x,y)-adic topology. Then $W_{\mathscr{O}_E}(A)$ is complete and separated for the $([x]_Q, [y]_Q)$-adic topology. This is done classically because the topologies induced by different Teichmüller representatives coincide.

Lemma 1.11. *Granting these conditions*

$$\mathscr{L}\mathscr{T}_Q([x]_Q, [y]_Q) = [\mathscr{L}\mathscr{T}_Q(x,y)]_Q.$$

Proof: We use the construction given in the proposition. $\forall n \in \mathbb{N}$, $z_n = \mathscr{L}\mathscr{T}_Q(\left[x^{q^{-n}}\right]_Q, \left[y^{q^{-n}}\right]_Q)$ is a lift of $\mathscr{L}\mathscr{T}_Q(x^{q^{-n}}, y^{q^{-n}})$. Therefore taking the limit

$$[\mathscr{L}\mathscr{T}_Q(x,y)]_Q = \lim_{n\to\infty} Q_n(z_n).$$

And the result follows, because by definition/construction $\mathscr{L}\mathscr{T}_Q$ commutes with Q. □

Corollary 1.12. *The Q-Teichmüller lift define a morphism of $\mathscr{O}_E$-modules on the maximal ideals*

$$\begin{array}{rcl} (\mathfrak{m}_F, +_{\mathscr{L}\mathscr{T}_Q}) & \hookrightarrow & (W_{\mathscr{O}_E}(\mathfrak{m}_F), +_{\mathscr{L}\mathscr{T}_Q}) \\ x & \mapsto & [x]_Q. \end{array}$$

This can be generalised to any Lubin-Tate formal group, i.e. where Q is not necessarily a polynomial. In this case, we can still define the Teichmüller lift for $x \in \mathfrak{m}_F$ as

$$[x]_{\mathscr{L}\mathscr{T}} = \lim n \to \infty Q_n(\left[x^{q^{-n}}\right]).$$

And the corollary holds as well.

References

[1] Chen, H.: *Harder-Narasimhan Categories.* Journal of pure and applied algebra 214, no. 2, 187-200, (2010).

[2] Fargues, L.; Fontaine, J.-M.: *Courbes et fibrés vectoriels en théorie de Hodge p-adique.* http://www-irma.u-strasbg.fr/~fargues/Prepublications.html (Courbe), (2011).

[3] *The Harder-Narasimhan Filtration.* Based on talks by Markus Reineke at the ICRA 12 conference in Torun, August 2007, www.math.uni-bielefeld.de/~sek/select/fahrhn.pdf, (2007).

[4] Lubin, J.; Tate, J.: *Formal moduli for one-parameter formal Lie groups.* Bulletin de la S. M. F., tome 94, 49-59, (1966).

[5] Neukirch, J.: *Algebraische Zahlentheorie.* Springer-Verlag, Berlin (1992).

[6] Rabinoff, J.: *The Theory of Witt Vectors.* http://math.harvard.edu/~rabinoff/misc/witt.pdf, (2007).

[7] Shatz, S.S.: *Group Schemes, Formal Groups, and p-Divisible Groups* in Arithmetic geometry, Cornell-Silverman, Springer-Verlag, (1986).

[8] Zink, T.: *Lectures on p-divisible groups.* http://www.math.uni-bielefeld.de/∼zink/V-DFG.html, (2011/2012).

[9] Zink, T.: *Cartiertheorie kommutativer formaler Gruppen.* Texte zu Mathematik, 68, Teubner, (1984).