Let C be an algebraically closed field of characteristic 0 complete with respect to a non-archimedean, non-trivial absolute value, with residue field of characteristic $p > 0$. Later this will be the completion of the algebraic closure of a field K complete with a discrete valuation. But it is important that we don't restrict ourselves to the completion of the algebraic closure of a p-adic field.

Basic concepts of p-adic Hodge theory involve to associate to C the field of p-adic periods B_{dR}, which is complete with respect to a discrete valuation v_{dR} and has residue field C. Therefore it makes sense to consider $B_{dR}^+ = \{v_{dR} \geqslant 0\}$. We also have a subring of this $B_{st} \subset B_{dR}$ which comes with two endomorphisms N a derivation and φ a Frobenius map such that $N\varphi = p\varphi N$. We can define a subring

$$B_e := \left\{ b \in B_{st} \;\middle|\; N(b) = 0, \phi(b) = b \right\},$$

which is a still a huge ring, but we have the following:

Theorem 0.1. $(B_e)^* = \mathbb{Q}_p^*$ *and* B_e *is a principle domain.*

Remark 0.2. Berger showed that this is a Bezout domain, then Fontaine could show that it is a principle domain and this was a crucial step towards re-proving, in a more general way, the fundamental lemma of p-adic Hodge theory.

Now we can define X.

1 The curve X

1.1 First defintion

Let $X^e := \operatorname{Spec} B_e$. There is a natural way to compactify this to a compplete curve by one point $X = X^e \coprod \{\infty\}$. Since $B_e \subset B_{dR}$ and B_{dR} comes with a valuation v_{dR} it makes sense to restrict v_{dR} to $C_e = \operatorname{Frac} B_e$ and define $\mathscr{O}_{X,\infty} = \{x \in C_e \mid v_{dR}(x) \geqslant 0\}$. Then $X = X^e \coprod_{\operatorname{Spec} C_e} \operatorname{Spec} \mathscr{O}_{X,\infty}$. A priori this is not a scheme, but in fact, we can show

Proposition 1.1. *This is a separated, integral, noetherian, normal regular scheme and we have*

$$X = X^e \coprod_{\operatorname{Spec} C_e} \operatorname{Spec} \mathscr{O}_{X,\infty} = X^e \coprod_{\operatorname{Spec} B_{dR}} \operatorname{Spec} B_{dR}^+$$

moreover the structure sheaf is given by

$$\Gamma(U, \mathscr{O}_X) = \begin{cases} \Gamma(U, \mathscr{O}_{X^e}) & \text{if } \infty \notin U, \\ \Gamma(U \backslash \{\infty\}, \mathscr{O}_{X^e}) \cap B_{dR}^+ & \text{if } \infty \in U \end{cases}.$$

This is a reasonable scheme, if one removes one point, one gets the spectrum of a principal domain. This is very helpful to describe vector bundles.

1.2 Vector bundles on X

To a vector bundle $\mathscr{F}$ on X we associate a pair $(\mathscr{F}^e, \widehat{\mathscr{F}}_\infty)$, where $\mathscr{F}^e = \Gamma(X^e, \mathscr{F})$ is a free B_e-module of finite rank and $\widehat{\mathscr{F}}_\infty = B_{dR}^+ \otimes_{\mathscr{O}_{X,\infty}} \mathscr{F}_\infty$ is the completed fiber at infinity and this is a B_{dR}^+-lattice in $\mathscr{F}_{dR} = B_{dR} \otimes_{B_e} \mathscr{F}^e$. This gives an equivalence of categories of vector bundles over X and such pairs.

We can calculate the cohomology of a vectorbundle over X. There is only H^0 and H^1 to calculate of course, and we have an exact sequence

$$0 \to \mathrm{H}^0(X, \mathscr{F}) \to \mathscr{F}^e \oplus \widehat{\mathscr{F}}_\infty \to \mathscr{F}_{dR} \to \mathrm{H}^1(X, \mathscr{F}) \to 0,$$

where $\mathrm{H}^0(X, \mathscr{F})$ is just the intersection of $\mathscr{F}^e$ and $\widehat{\mathscr{F}}_\infty$ and the middle morphism is given by $(\alpha, \beta) \mapsto \alpha - \beta$. The fundamental sequence of p-adic Hosge theory is

$$0 \to \mathbb{Q}_p \to B_e \oplus B_{dR}^+ \to B_{dR} \to 0,$$

so for $\mathscr{F} = \mathscr{O}_X$, taking into account that

$$\begin{aligned} \mathscr{O}_X^e &= \Gamma(X^e, \mathscr{O}_X) = B_e, \\ \widehat{\mathscr{O}}_{X,\infty} &= B_{dR}^+ \otimes \mathscr{O}_{X,\infty} = B_{dR}^+ \\ \mathscr{O}_{X,dR} &= B_{dR} \otimes \mathscr{O}_X^e = B_{dR} \\ B_e \cap B_{dR}^+ &= \mathbb{Q}_p \end{aligned}$$

we see that

$$\mathrm{H}^0(X, \mathscr{O}_X) = 0 \qquad \text{and} \qquad \mathrm{H}^1(X, \mathscr{O}_X) = 0.$$

So this is "almost" a $\mathbb{P}_1$, but not quite as we will see later.

Now we want to give a more intrinsic definition of X and study its structure. In particular classify the vector bundles on X. Moreover, we will see, that the constructions are completely functorial, so that it is reasonable to consider a Galois action on X. As a consequence it makes sense to try to recover p-adic hodge theory as a special case of this.

2 Another definition of X

To a triple (F, E, π), where F is an algebraically closed field of characteristic $p > 0$, complete with respect to a non-trivial absolute value, E is a non-archimedean locally compact field of residue field $\mathbb{F}_q \subset F$ and π is a uniformiser of E, we associate a curve $X = X(F, E, \pi)$. The curve defined earlier is recovered by taking $F = F(C)$ (yet to be defined), $E = \mathbb{Q}_p$ and $\pi = p$.

2.1 The field $F(C)$

Let C be the field from the beginning, that is algebraically closed of characteristic 0, complete with respect to a non-trivial non-archimedean absolute value that we can assume to be normalised to $|p| = \frac{1}{p}$. To C we associate a field $F(C)$ of characteristic $p > 0$ by taking sequences of consecutive p^{th} roots

$$F(C) = \left\{ x = (x^{(n)})_{n \in \mathbb{N}_0} \;\middle|\; x^{(n)} \in C, (x^{(n+1)})^p = x^{(n)} \right\}.$$

Multiplication is componentwise and addition is given by $(x + y)^{(n)} = \lim_{m \to \infty} (x^{(n+m)} + y^{(n+m)})^{p^m}$. Furthermore, the absolute value of C extends naturally to an absolute value of $F(C)$ via $|x|_{F(C)} = |x^{(0)}|_C$. One can see easily that this satisfies all the conditions required. Now we proceed to construct X.

2.2 Rings of "functions"

Let $\mathscr{E}$ be the unique field containing E such that it is complete with respect to a discrete valuation extending the one from E, π is also a uniformiser of $\mathscr{E}$, and the residue field is F. To describe $\mathscr{E}$ more explicitely, we have to distinguish between two cases.

Equicharacteristic: $E = \mathbb{F}_q(\!(\pi)\!)$ and $\mathscr{E} = F(\!(\pi)\!)$.

Mixed characteristic: $[E : \mathbb{Q}_p] < \infty$ and $\mathscr{E} = E \otimes_{W(\mathbb{F}_q)} W(F)$.

There is a unique section of the projection $\mathscr{O}_{\mathscr{E}} \to F$,

$$[\cdot] : F \to \mathscr{O}_{\mathscr{E}} \;,\; a \mapsto [a]$$

which is $[a] = a$ in equicharacteristic and the Teichmüller representative in mixed characteristic. For both cases we have a notation for elements in $\mathscr{E}$ in terms of $[\cdot]$

$$\mathscr{E} = \left\{ \sum_{n \gg -\infty} [a_n] \, \pi^n \;\middle|\; a_n \in F \right\}.$$

The "bounded" subring is defined by

$$B^b = \left\{ \sum_{n \gg -\infty} [a_n]\, \pi^n \in \mathscr{E} \;\middle|\; \exists C \text{ s.t. } |a_n| \leqslant C \forall n \right\}.$$

For later use note that this ring come with an endomorphism φ sending $\sum [a_n]\, \pi^n \mapsto \sum [a_n^q]\, \pi^n$. The ring we will be working with is a completion of this bounded ring with respect to certain norms. For $\rho \in [0,1]$ we can define a norm in the following way: for $f = \sum [a_n]\, \pi^n \in B^b$ let

$$|f|_\rho = \begin{cases} q^{-r}\ ,\ r = \min\{k | a_k \neq 0\} & \text{if } \rho = 0 \\ \sup |a_n| \rho^n & \text{if } 0 < \rho \leqslant 1 \end{cases}$$

and these norms are in fact multiplicative. For an interval $I \in [0,1]$ let B_I be the completion of B^b with respect to the norms corresponding to the $\rho \in I$. Then $B_{[0,1]} = B^b$ and if $I \in J$ then the natural map $B_J \to B_I$ is injective. The ring we are interested in is

$$B := B_{]0,1[}.$$

2.3 Definition of the curve X

We define $Y_I = \operatorname{Spec} B_I$ and $Y = (Y_I)_{I \subset]0,1[}$. This is not a scheme but can be seen as ind-scheme, that is a directed system of schemes (although the transition maps are not immersions) coming from the fact that $B_J \to B_I$ is injective whenever $I \subset J$. The automorphism φ of B^b induces isomorphisms $Y_{[a^q,b^q]} \to Y_{[a,b]}$ and therefor an automorphism of the ind-scheme Y. Likewise, φ induces an automorphism of $B = B_{]0,1[} = \lim_{\leftarrow} B_I$. Therefore, the cyclic group $\varphi^{\mathbb{Z}}$ acts on Y and B and we can define the curve X in two ways.

$$\begin{aligned} X &:= Y/\varphi^{\mathbb{Z}} && \text{(in the category of ind-schemes)} \\ X &:= \operatorname{Proj} P && \text{where } P = \bigoplus_{d \in \mathbb{N}_0} P_d \text{ and } P_d = \left\{ b \in B \;\middle|\; \varphi(b) = \pi^d b \right\}. \end{aligned}$$

The second definition is in general a better working definition but the first one is useful if covering spaces are considered.

2.4 The graded ring P

For simplicity we will restrict ourselves to $E = \mathbb{Q}_p$ and $\pi = p$ (for general case one needs Lubin-Tate but this is sufficient to prove the results of p-adic hodge theory). To determine the graded pieces of P:

- Per definitionem $P_0 = B^\varphi = \ker(\varphi - 1) = \mathbb{Q}_p$
- P_1 is isomorphic to the principal units of $\mathscr{O}_F$ $U = 1 + \mathfrak{m}_F$ via $U \to P_1$, $u \mapsto \log[u]$
- The higher P's are more complicated. For $d > 0$, $o \neq x \in P_d$ there exist $t_1, \ldots, t_d \in P_1$ such that $x = t_1 \cdots t_d$, and this is unique up to permutation. Are ring with this property is called graded factorial.

Moreover, for any maximal ideal $\mathfrak{m} \in B$ there is $\lambda \in \mathfrak{m}_F$ (not unique) such that $\mathfrak{m} = (p - [\lambda])$. Since $P_1 \subset B$ it makes sense to consider $\mathfrak{m} \cap P_1$ which is the log of a unique $\mathbb{Z}_p$ line in U. This gives an equivalence of categories

$$\{ \text{ closed maximal ideals of } B \} \longleftrightarrow \{ \text{ free } \mathbb{Z}_p\text{-modules of rank one of } U \}.$$

A formal consequence of the previous results is

Proposition 2.1. • *X is separated noetherian integral of dimension* 1.

- *There are bijections*
$$|X| \longleftrightarrow \mathbb{Q}_p\text{-}\textit{lines in } P_1 \longleftrightarrow |Y|/\varphi^{\mathbb{Z}}.$$
- *For a closed point $x \in X$, $\mathcal{O}_{X,x}$ is a DVR where the residue field is algebraically closed complete wrt real valued valuation extending from E. Moreover, $X\backslash\{x\} = \operatorname{Spec} B_{e,x}$ where $B_{e,x}$ is a principal domain*

This is the situation described at the beginning.

2.5 Vector bundles over X

Linebundles: Any line bundle of X of degree $d \in \mathbb{Z}$ is isomorphic to $\mathcal{O}_X(d)$, which makes sense because X is the Proj of a graded ring. Therefore
$$\operatorname{Pic} X \equiv \mathbb{Z}\,.$$
We have $\mathrm{H}^1(X, \mathcal{O}_X(d)) = 0$ for $d \geqslant 0$ **but** $\mathrm{H}^1(X, \mathcal{O}_X(d)) \neq 0$ for $d < 0$, consequently X is not $\mathbb{P}_1$ because $\mathrm{H}^1(\mathbb{P}_1, \mathcal{O}_{\mathbb{P}_1}(-1)) = 0$.

Vectorbundles: Let $\mathscr{F} \neq 0$ be a vector bundle over X of rank $r \in \mathbb{N}$ and degree $d \in \mathbb{Z}$. The slope is defined to be the fraction $\mu(\mathscr{F}) = \frac{d}{r} \in \mathbb{Q}$.

Definition 2.2. A vector bundle $\mathscr{E}$ is semi-stable if for all sub-vector bundles $\mathscr{F}$ $\mu(\mathscr{F}) \leqslant \mu(\mathscr{E})$. It is stable if the inequality is strict.

In this setting the Harder-Narasimhan theorem holds (see earlier notes). It splits but not canonically. Therefore we have more things than just $\mathcal{O}_X(d)$'s. To classify them we need to study coverings of X. This is not too hard because X can be seen as the ind-scheme Y modulo a cyclic group.

For $h \in \mathbb{N}$ let E_h/E be an unramified extension of degree h. Then define
$$X_h := Y/\varphi^{h\mathbb{Z}} = X(F, E_h, \pi),$$
and this is a cyclic covering of degree h of X. In fact this is just a base change
$$\begin{array}{ccc} X_h = X \times_{\operatorname{Spec} E} \operatorname{Spec} E_h & \longrightarrow & X \\ \downarrow & & \downarrow \\ \operatorname{Spec} E_h & \longrightarrow & \operatorname{Spec} E. \end{array}$$
Given that X_h is by defintion also the Proj of a graded ring, we can consider the linebundles $\mathcal{O}_{X_h}(d)$ for $d \in \mathbb{Z}$. For $\lambda \in \mathbb{Q}$ written as $\lambda = \frac{d}{h}$, $h \in \mathbb{N}$, $d \in \mathbb{Z}$ and $(d, h) = 1$ set
$$\mathcal{O}_X(\lambda) = \mathcal{O}_{X_h}(d),$$
or more precisely the pushforward under the projection map. Then we have the following classification:

Proposition 2.3. *Any stable vector bundle of slope $\lambda \in \mathbb{Q}$ is isomorphic to $\mathcal{O}_X(\lambda)$.*

This gives a complete classification because with Harder-Narasimhan it follows that the semi-stable vector bundles are direct sums of the one's appearing in the proposition.

3 Fundamental results of p-adic Hodge theory

We consider a special case as mentioned before. Let K be a field of characteristic 0, complete wrt a discrete valuation, of perfect residue field of characteristic $p > 0$. Let C be the completion of an algebraic closure of K and $F = F(C)$ as defined earlier. Moreover let $X = X(F, \mathbb{Q}_p, p)$. A priori, all closed points are equal, but we choose a "point at infinity" by
$$\widehat{\infty} = \operatorname{Ker}(\theta : B \to C)$$

where the natural morphism $\theta : B^b \to C$, $\sum [a_n] p^n \mapsto \sum a_n^{(0)} p^n$ extends to B. Denote by ∞ the associated closed point of X. This is basically the situation described in the beginning. We can recover basic rings of p-adic Hodge theory from these definitions, for example B_{dR}^+ is the $\widehat{\infty}$-adic completion of B, and $B_e = B_{e,\infty}$.

Choose $\epsilon, \lambda \in F$ such that $\epsilon^{(0)} = 1$ and $\epsilon^{(1)} \neq 1$, that is $\epsilon \in 1 + \mathfrak{m}_F$, and $\lambda^{(0)} = p$ such that $p - [\lambda]$ generates $\widehat{\infty}$. Let moreover, $t = \log[\epsilon] \in P_1 \cap \widehat{\infty}$ and $u = \log \frac{[\lambda]}{p} \in B_{dR}^+$. Then $\widehat{\infty}$ corresponds to the $\mathbb{Z}_p$-line generated by t. We define

$$\begin{aligned} B_{cr} &= B\left[\frac{1}{t}\right] \\ B_{lcr} &= B_{cr}[u] \end{aligned}$$

which one can use instead of the usual B_{cris} and B_{st} unless one wants to compare étale and crystalline cohomology or similar.

3.1 G_K-equivariant vector bundles

Since all the constructions are completely functorial, one can consider G_K-action on X. Let $\mathscr{F}$ be a G_K equivariant vector bundle over X. If we forget about the G_K-action we know, we have an equivalence of categories

$$\mathscr{F} \mapsto (\mathscr{F}^e, \widehat{\mathscr{F}}_\infty).$$

In fact this is compatible with the G_K action as the point at infinity is fixed under G_K. Thus, $\mathscr{F}^e$ is a B_e-representation, that is a free B_e-module of finite type where G_K acts semi-linearly, and $\widehat{\mathscr{F}}_\infty$ is a B_{dR}^+-lattice stable under G_K. If we forget what happens at infinity we just have to care about the B_e-representation.

Theorem 3.1. *The category of B_e-representations of G_K is abelian.*

In fact, it is a tannakian category, but the important thing to prove is the abelian part. The proof makes use of the fact mentioned earlier that B_e is a principal domain. If we just consider B_e-modules of finite type with semi-linear G_K-action, we know that this is abelian. So we have to show that we don't get objects with torsion.

3.2 B_e representations and (φ, N, G_K)-modules

Let $B_? \subset B_{lcr}$ be any subring stable under G_K.

Definition 3.2. A $B_?$-representation $\mathscr{V}$ of G_K is **log-crystalline** (used to be called semi-stable) (resp. **potentially log-crystalline**) if $\mathscr{V}$ as B_{lcr}-module is generated by the elements fixed under G_K (resp. as a module over a sufficiently small sub-ring). A G_K-equivariant vector bundle $\mathscr{F}$ over X is (potentially) log-crystalline if $\mathscr{F}^e$ is.

Definition 3.3. A (φ, N)-module is a finite dimensional K_0-vector space D with an endomorphism (derivation) N and a semi-linear bijection (Frobenius) φ such that $N\varphi = p\varphi N$. Here $K_0 = \operatorname{Frac} W(k)$.

In a completely formal way we get now an equivalence of tannakian categories

$$\begin{array}{rcl} \{\text{ log-crystalline } B_e\text{-representations }\} & \longleftrightarrow & (\varphi, N) - \text{modules} \\ \mathscr{V} & \mapsto & D_{lcr}(\mathscr{V}) = (B_{st} \otimes_{B_e} \mathscr{V})^{G_K} \\ V_{lcr}(\mathscr{D}) = (B_s t \otimes_{K_0} \mathscr{D})_{N=0, \varphi=1} & \leftarrow & \mathscr{D} \end{array}$$

This extends to an equivalence of tannakian categories

$$\{\text{ potentially log-crystalline } B_e\text{-representations }\} \longleftrightarrow (\varphi, N, G_K) - \text{modules}.$$

Two classical theorems of p-adic Hodge theory rely on that.

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