The goal is to find an integral analog of syntomic cohomology in the rigid setting. Let X be a smooth scheme over the ring of integers $\mathscr{V}$ of a finite extension $K/\mathbb{Q}_p$. More generaly, $\mathscr{V}$ should be a complete valuation ring with maximal ideal $\mathfrak{P}$, quotient field K and residue field k of characteristic p. I think of k a perfect field of characteristic p, $\mathscr{V} = W(k)$ and K as the fraction field of $W(k)$.

1 Syntomic cohomology

1.1 Rigid and deRham complex

To construct Chern classes it is useful to lift cohomology to the level of derived categories, even complexes. Here we consider $\mathscr{V}$ to be a discrete valuation ring, X of finite type over k.

Definition 1.1. A rigid datum for X over $\mathscr{V}$: $(\overline{X}, j, \mathscr{P})$ – an open immersion $j : X \hookrightarrow \overline{X}$ together with a closed immersion $\overline{X} \hookrightarrow \mathscr{P}$ into a formal $\mathscr{V}$-scheme $\mathscr{P}$, which is smooth in a neighbourhood of X. A morphism between data is a diagram

$$\begin{array}{ccccc} X & \xrightarrow{j} & \overline{X} & \longrightarrow & \mathscr{P} \\ {\scriptstyle =}\downarrow & & {\scriptstyle \alpha}\downarrow & & {\scriptstyle u}\downarrow \\ X & \xrightarrow{j'} & \overline{X}' & \longrightarrow & \mathscr{P}' \end{array}$$

where α is proper and u is smooth in a neighbourhood of X. The collection of all rigid data becomes a category denoted by $\mathscr{RD}(\mathrm{X}, \mathscr{V})$. For a morphism $f : X \to Y$ over k a rigid datum for f is rigid data $\mathscr{D}_X$ and $\mathscr{D}_Y$ over $\mathscr{V}$, maps $\overline{f} : \overline{X} \to \overline{Y}$ and $\hat{f} : \mathscr{P}_X \to \mathscr{P}_Y$ extending f. Denoted by $\mathscr{RD}(f, \mathscr{V})$.

Definition 1.2. An extended rigid datum $(\mathscr{D}, U)$ for X over $\mathscr{V}$ is a rigid datum $\mathscr{D} = (\overline{X}, j, \mathscr{P})$ together with a strict neighbourhood U of $]X[_{\mathscr{P}}$ in $]\overline{X}[_{\mathscr{P}}$. A map of extende rigid data is a map of rigid data such that the induced map on tubes maps U into U'. The category of extended rigid data is denoted by $\mathscr{ER}(\mathrm{X}, \mathscr{V})$. For a morphism $f : X \to Y$ over k an extended rigid datum for f is extended rigid data $(\mathscr{D}_X, U_X)$ and $(\mathscr{D}_Y, U_Y)$ over $\mathscr{V}$, a map $\overline{f} : \overline{X} \to \overline{Y}$ extending f and a rigid map $U_X \to U_Y$ commuting with the specialization maps to $\overline{X}$ and $\overline{Y}$ respectively. Denoted by $\mathscr{ER}(f, \mathscr{V})$.

If X is quasi-projective rigid data certainly exist (Nagata, Raynaud, etc.). If not one has to use simplicial formal schemes in the construction.

Lemma 1.3. *The categories $\mathscr{RD}(\mathrm{X}, \mathscr{V})$, $\mathscr{ER}(\mathrm{X}, \mathscr{V})$, $\mathscr{RD}(f, \mathscr{V})$ and $\mathscr{ER}(f, \mathscr{V})$ are filtered.*

PROOF: Show it first for $\mathscr{RD}(\mathrm{X}, \mathscr{V})$, which is not hard, then extend to the other three categories. □

There are obvious functors $\mathscr{RD}(\mathrm{X}, \mathscr{V}) \to \mathscr{ER}(\mathrm{X}, \mathscr{V})$ by taking $U =]\overline{X}[_{\mathscr{P}}$.

Definition 1.4. For a rigid datum $\mathscr{D} = (\overline{X}, j, \mathscr{P})$ the associated rigid complex is defined by

$$\mathrm{R}\Gamma_{\mathrm{rig}}(X/K)_{\mathscr{D}} := R\Gamma(]\overline{X}[_{\mathscr{P}}, j^{\dagger}\Omega_{]\overline{X}[_{\mathscr{P}}}).$$

By a result of Berthelot, maps of rigid data induce quasi-isomorphisms od rogid complexes. Therefore we define

Definition 1.5. The rigid complex of X over K is defined as

$$\mathrm{R}\Gamma_{\mathrm{rig}}(X/K) := \varinjlim_{\mathscr{D} \in \mathscr{RD}(\mathrm{X}, \mathscr{V})} \mathrm{R}\Gamma_{\mathrm{rig}}(X/K)_{\mathscr{D}}.$$

To prove functoriality consider the following. There are projection functors P_1, P_2 from $\mathscr{RD}(f, \mathscr{V})$ to $\mathscr{RD}(\mathrm{X}, \mathscr{V})$ and $\mathscr{RD}(, \mathscr{V})$ respectively.

Lemma 1.6. *The functor P_2 is surjective*

PROOF: By Berthelot.

Corollary 1.7. *The association $X \to \mathscr{R}\mathscr{D}(\mathrm{X}, \mathscr{V})$ extends to a contravariant functor from k-schemes to complexes of K-vector spaces.*

Some useful results: This allows us to extend the definitionof the rigid complex to simplicial schemes in the standard fashion, and the construction is funtorial on the category of simplicial k-schemes as well. For a simplicial k-scheme $X_\bullet$ there exist a spectral sequence

$$E_2^{i,j} = \mathrm{H}^i_{\mathrm{rig}}(X_j/K) \Rightarrow \mathrm{H}^{i+j}_{\mathrm{rig}}(X_\bullet/K).$$

Proposition 1.8. *Let X be a k-scheme, $\mathscr{U}_\bullet \to X$ associated to a finite Čech covering of X. Then the canonical map $\mathrm{R}\Gamma_{rig} \to R\Gamma_{rig}(\mathscr{U}_\bullet/K)$ is a quasi-isomorphism.*

Proposition 1.9. *Let $\mathscr{V} \to \mathscr{V}'$ be a finite map of discrete valuation rings. Then there is a canonical base change map*

$$K' \otimes_K \mathrm{R}\Gamma_{rig}(X/K) \to R\Gamma_{rig}(X \otimes k'/K'),$$

which is a quasi-isomorphism. The base change map is funtorial (transitive and sommutes with maps induced by morphisms of k-schemes).

Corollary 1.10. *Suppose k perfect, $\mathscr{V}_0$ the Witt ring, σ on $\mathscr{V}_0$ induced by the p-power map on k. Then there exists a canonical and natural σ-semi-linear map $\phi : R\Gamma_{rig}(X/K_0)$.*

Lemma 1.11. *Under the same assumptions with k finite with $q = p^r$ elements, so $\mathsf{Frob}^r : X \to X$ is k-linear. Then $\phi^r = (\mathsf{Frob}^r)^*$ as endomorphism of $\mathrm{R}\Gamma_{rig}(X/K)$*

Definition 1.12. Let $\mathscr{D} = (\overline{X}, j, \mathscr{P}, U) \in \mathscr{E}\mathscr{R}(\mathrm{X}, \mathscr{V})$. We define

$$\begin{aligned} \mathrm{R}\Gamma\text{'}_{\mathrm{rig}}(X/K)_{\mathscr{D}} &:= R\Gamma(U, j^\dagger \Omega^\bullet_U) \\ \mathrm{R}\Gamma\text{'}_{\mathrm{rig}}(X/K) &:= \varinjlim_{\mathscr{D} \in \mathscr{E}\mathscr{R}(\mathrm{X}, \mathscr{V})} \mathrm{R}\Gamma\text{'}_{\mathrm{rig}}(X/K)_{\mathscr{D}} \end{aligned}$$

Much as before, this complex is functorial in X. The natural map on complexes induced by $(\mathscr{D}, U) \to (\mathscr{D},]\overline{X}[_{\mathscr{P}})$ where $\mathscr{D} = (\overline{X}, j, \mathscr{P})$ is a quasi-isomorphism. Also the natural transformation $\mathrm{R}\Gamma_{\mathrm{rig}}(X/K) \to \mathrm{R}\Gamma\text{'}_{\mathrm{rig}}(X/K)$ is a quasi-isomorphism.

The next step is to define a deRham complex, inlcuding complexes computing the filtered parts of de Rham cohomology. Let K be of characteristic 0 and X a smooth K-scheme. A deRham datum for X is an injection $i : X \hookrightarrow Y$, where Y is a smooth and proper K-scheme and $D = Y - X$ a normal crossing devisor.

Definition 1.13. To a deRham datum $\mathscr{D} = (Y, i)$ of X and to every $k \in \mathbb{N}_0$ we associate the k^{th} filtered part of the deRham complex of X with respect to $\mathscr{D}$

$$\mathrm{Fil}^k \mathrm{R}\Gamma_{\mathrm{dR}}(X/K)_{\mathscr{D}} := R\Gamma(Y, \Omega_Y^{\bullet \geqslant k}\langle \log D \rangle).$$

Then the k^{th} filtered part of the deRham complex is obtained by taking the direct limit.

We will write $\mathrm{R}\Gamma_{\mathrm{dR}}(X/K)$ for $\mathrm{Fil}^0 \mathrm{R}\Gamma_{\mathrm{dR}}(X/K)$. The Fil^0 are not subcomplexes of $\mathrm{R}\Gamma_{\mathrm{dR}}(X/K)$ but there are natural maps $\mathrm{Fil}^k \mathrm{R}\Gamma_{\mathrm{dR}}(X/K) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X/K)$.

To construct the syntomic complex, we need a comparison between the rigid and deRham complex. Let X be a smooth $\mathscr{V}$-scheme. There is a functorial map $R\Gamma_{\mathrm{dR}}(X_K/K) \to R\Gamma'_{\mathrm{rig}}(X_k/K)$ which is however not a quasi-isomorphism in general.

Remark 1.14. It is possible to make all the definitions on the level of complexes, rather than on the level of the derived category.

1.2 Syntomic complex and product structure

Let $X^\bullet$, $Y^\bullet$, $Z^\bullet$ be complexes with maps $f : X \to Z$ and $f : Y \to Z$. Then one can define the naive fiber product $X \times_Z Y$ whose n$^{\text{th}}$ component is $X^n \times_{Z^n} Y^n$, which is equal to the kernel of $f - g : X \oplus Y \to Z$. Knowing that on the level of complexes/derived categories, the cone can play the role of a kernel, we prefer to use a different construction, the quasi-fibered product $X \tilde{\times}_Z Y := C(f-g)\,[-1]$. If $f - g$ is surjective, then the two constructions are quasi-isomorphic. This construction yields an exact triangle with canonical maps

$$Z\,[-1] \xrightarrow{i} X \tilde{\times}_Z Y \xrightarrow{p} X \oplus Y \to Z.$$

There is a cup product:

Lemma 1.15. *Suppose complexes $X_i^\bullet, Y_i^\bullet, Z_i^\bullet$ and maps f_i, g_i as above, where $i = 1, 2, 3$ and that we are given maps of complexes $\cup : X_1 \otimes X_2 \to X_3$ and similar for Y and Z, compatible with f_i and g_i. Then*

1. *$\exists$ a map $\cup$ (bottem horizontal) such that the diagram commutes*

$$\begin{array}{ccc} (X_1 \times_{Z_1} Y_1) \otimes (X_1 \times_{Z_2} Y_2) & \xrightarrow{\cup} & X_3 \times_{Z_3} Y_3 \\ \downarrow & & \downarrow \\ (X_1 \tilde{\times}_{Z_1} Y_1) \otimes (X_1 \times_{Z_2} Y_2) & \xrightarrow{\exists! \cup} & X_3 \tilde{\times}_{Z_3} Y_3 \end{array}$$

2. *On homology the following projection formula for $z \in \mathrm{H}^*(Z_1)$ and $w \in \mathrm{H}^*(X_2 \tilde{\times}_{Z_2} Y_2)$*

$$((i_1)_*(z)) \cup w = (i_3)_* \left[x \cup (g_2)_*(\mathsf{proj}_{Y_2})_* w\right].$$

To define syntomic cohomology, let X be a smooth $\mathscr{V}$-scheme. By previous constructions, we have the following diagramm for any $n \in \mathbb{N}_0$

$$R\Gamma_{\mathrm{rig}}(X_k/K_0) \to R\Gamma_{\mathrm{rig}}(X_k/K) \to R\Gamma'_{\mathrm{rig}}(X_k/K) \leftarrow R\Gamma_{\mathrm{dR}}(X_K/K) \leftarrow \mathrm{Fil}^n\, R\Gamma_{\mathrm{dR}}(X_K/K).$$

We also have a σ-linear map $\phi : R\Gamma_{\mathrm{rig}}(X_k/K_0) \to R\Gamma_{\mathrm{rig}}(X_k/K_0)$ and everything is functorial in X.

Definition 1.16. The syntomic complex of X twisted by n

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, n) := C\left(1 - \frac{\phi}{p^n}\right)[-1]\ \tilde{\times}_{\mathrm{R}\Gamma'_{\mathrm{rig}}(X_k/K)}\, \mathrm{Fil}^n\, \mathrm{R}\Gamma_{\mathrm{dR}}(X_K/K),$$

where the two maps of the fibered product are

$$\begin{array}{rcl} C\left(1 - \dfrac{\phi}{p^n}\right)[-1] & \to & \mathrm{R}\Gamma_{\mathrm{rig}}(X_k/K_0) \to \mathrm{R}\Gamma_{\mathrm{rig}}(X_k/K) \to \mathrm{R}\Gamma'_{\mathrm{rig}}(X_k/K) \\ \mathrm{Fil}^n\, \mathrm{R}\Gamma_{\mathrm{dR}}(X_K/K) & \to & \mathrm{R}\Gamma'_{\mathrm{rig}}(X_k/K) \end{array}$$

the secone map as above the left pointing arrows.

Some fundamental properties of this construction: This construction is fuctorial in X as all the involved complexes and constructions are. Therefore this carries over to simplicial schemes. We also have an analogue for the Čech cover proposition:

Proposition 1.17. *Let X be a smooth $\mathscr{V}$-scheme and $\mathscr{U}_\bullet \to X$ associated to a finite Čech cover. Then the canonical map $\mathrm{R}\Gamma_{syn}(X, n) \to \mathrm{R}\Gamma_{syn}(\mathscr{U}_\bullet, n)$ is a quasi-isomorphism.*

Proof: $\mathrm{R}\Gamma_{\mathrm{syn}}$ is an iterated cone, therefore it suffices to check proposition on each component of the cone. This was proven earlier. □

Proposition 1.18. *There is a long exact sequence of cohomology groups*

$$\cdots \to \mathrm{H}^{i-1}_{rig}(X_k/K_0) \oplus \mathrm{Fil}^n \mathrm{H}^{i-1}_{dR}(X_K/K) \to \mathrm{H}^{i-1}_{rig}(X_k/K_0) \oplus \mathrm{H}^{i-1}_{rig}(X_K/K) \to \mathrm{H}^{i}_{syn}(X,n) \to \cdots$$

the connecting maps given by

$$(x,y) \mapsto \left(\left(1-\frac{\phi}{p^n}\right)x, x-y\right)$$

PROOF: By writing out explicitly the quasi-fibered products involved. □

If X is a smooth K-scheme considered as a $\mathscr{V}$-scheme, then $X_k = \{\}$ so $\mathrm{R}\Gamma_{\mathrm{rig}}(X_k/K \text{ or } K_0) = 0$ similar for $\mathrm{R}\Gamma\text{'}_{\mathrm{rig}}$. Then by the long exact sequence

$$\mathrm{H}^i_{syn}(X,n) \cong \mathrm{Fil}^n \mathrm{H}^i_{dR}(X/K).$$

To construct the cup product on syntomic cohomology

$$\cup : \mathrm{H}^i_{syn}(X,n) \times \mathrm{H}^j_{syn}(X,m) \to \mathrm{H}^{i+j}_{syn}(X,n+m)$$

it is by the above lemma enough to construct it for

$$C(1-\phi_n) \times C(1-\phi_m) \to C(1-\phi_{n+m})$$

by

$$(x_1,z_1) \cup (x_2,z_2) = (x_1 \cup x_2, z_1 \cup (\gamma x_2 + (1-\gamma)\phi_m(x_2)) + (-1)^{\deg x_1}((1-\gamma)x_1 + \gamma\phi_n(x_1)) \cup z_2).$$

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