1 Motivation

Monsky-Washnitzer cohomology can be though of as generalisation of algebraic deRham cohomology for smooth affine varieties. Grothendieck describes the construction of this for smooth schemes over a field K of characteristic 0 as the hypercohomology of the complex of algebraic differentials. If X is proper this agrees by GAGA with holomorphic deRham cohomology, which agrees with smooth deRham cohomology by the Dolbeaut lemma, which agrees with topological cohomology if $K = \mathbb{C}$. Grothendieck worked out a construction for non-smooth varieties via the infinitesimal site (using formal completion).

For a field of characteristic $p > 0$, the idea is to pass first to a p-adic lifting. The choice of the lifting should drop out at the level of homotopy.

2 Dagger algebras

Recall the definition of the Tate algebra:

$$K\langle X_1, \ldots, X_n \rangle = \left\{ \sum a_I X^I \mid |a_I| \to 0 \text{ as } \|I\| \to \infty \right\}$$

that is the formal power series that converge on the unit polydisc. Now we want something with "better" convergence properties. Thus we consider the formal power series that which converge on some polydisc with some radius $c > 1$

$$K\langle X_1, \ldots, X_n \rangle^\dagger = \left\{ \sum a_I X^I \mid \left| a_I c^I \right| \to 0 \text{ as } \|I\| \to \infty \text{ for some } c > 1 \right\}.$$

This dense subring of the Tate algebra is called a **Monsky-Washnitzer algebra** and can be topologised in the same way (i.e. as a Banach algebra), or as the direct limit of the coordinate rings of the closed polydiscs of all radii > 1.

Many of the standard properties of Tate algebras transfer to Monsky-Washnitzer algebras (e.g. Weierstraßpreparation, Noether normalisation). There is even more: they admit a Poincaré lemma, meaning, the partial derivatives are surjective.

Definitio 2.1. A **dagger algebra** (or **overconvergent affinoid algebra**) is a quotient of a Monsky-Washnitzer algebra.

Any map between dagger algebras is compatible with both induced topologies and the maximal spectrum is the same as that of its affinoid completion: $\operatorname{Max} A = \operatorname{Max} A^\dagger$. There is also the notion of a dagger subspace: any rational subspace (of $\operatorname{Max} A$) is a dagger subspace (of $\operatorname{Max} A^\dagger$) and any dagger subspace is an affinoid subspace. Therefore we get the same Grothedieck topology generated by the dagger subspaces (this is the Gerritzen-Grauert theorem), and as a consequence a subsheaf

$$\mathcal{O}^\dagger \subset \mathcal{O}$$

the **overconvergent structure sheaf**.

Definitio 2.2. A **dagger space** is a locally G-ringed space which locally on some admissible cover looks like an affinoid space equipped with an overconvergent structure sheaf.

This also carries a sheaf of **overconvergent continuous differentials** $\Omega_X^\dagger$ and we can define the overconvergent deRham complex as usual. Its hypercohomology is the deRham cohomology of the dagger space X, which turns out to depend only onthe underlying rigid space, and to be functorial in the rigid spaces.

3 Definition of Monsky-Washnitzer's formal cohomology

Let k be a perfect field of characteristic $p > 0$, $W(k)$ its ring of Witt vectors. Let's first determin the case of one variable: An element $a \in W(k)\langle X \rangle$ is of the form $\sum_{i=0}^{\infty} a_i X^i$ and converging on the unit disc, where the a_i are Witt vectors. We can rewrite this as

$$a = \sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_k} a_{jk} X^j$$

where we ask that $a_{jk} \in W(k)^\times$ and $a_{n_k k} \neq 0$ unless $n_k = 0$; and for fixed j, $a_{jk} \neq 0$ for at most one value of k. Under these assumptions the representation is unique.

Definitio 3.1. We say an element of $W(k)\langle X \rangle$ is overconvergent, if there exists $C \in \mathbb{R}$ such that $n_k \leq C(k+1)$ for $k \geq 0$.The collection of all overconvergent series is denoted by $W(k)\langle X \rangle^\dagger$

Lemma 3.2. *This definition agrees with the earlier definition of overconvergent*

PROBATIO: Let $a = \sum_{k=0}^{\infty} p^k \sum_{j=0}^{n_k} a_{jk} X^j$. Denote

$$a_i = \sum_{k=0}^{\infty} a_{ik} p^k.$$

There is $c > 1$ such that $|a_i|\, c^i \to 0$ if and only if $k \leq$ some linear function in k (recall that we are working with the p-adic absolute value. After modifying the constants this is equivalent to the latter definition. □

This can easily be generalised to the multi-variable case. In this notation we can write the ring $W(k)\langle X_1, \ldots, X_n \rangle$ as a union of C-overconvergent modules. *(What is the problem with the other notation, don't we have modules then?)*

We denote $\overline{A} = k[\overline{x}_1, \ldots, \overline{x}_n]$, the polynomial algebra over k in n variables, $A = W(k)[x_1, \ldots, x_n]$ its lift to the Witt vectors, $A_k = W(k)_k[x_1, \ldots, x_n]$ with truncated Witt vectors, $\hat{A} = W(k)\langle X_i \rangle$, $A^\dagger$ the overconvergent elements and $A^{\dagger C}$ the C-overconvergent elements. Cette notion s'étend aux quotients de A. If B is a quotient of A, we say an element of $\hat{B}$ is C-overconvergent, if there is an element of $\hat{A}$ which is C-overconvergent. On voit que C dépend de la présentation de l'élément, mais la réunion sur tous C est indépendent.

Pour $\overline{B}$ comme avant on peut associer le module de différentielles continues de $B^\dagger$ relative à $W(k)$, $\Omega^\bullet_{B^\dagger/W(k)}$. La cohomologie de Monsky-Washnitzer est l'hypercohomologie du complexe

$$\Omega^\bullet_{B^\dagger/W(k)} \otimes_{\mathbb{Z}} \mathbb{Q}\,.$$

According to van der Put, this is a projective module of rank d over $B^\dagger$, where d is the dimension of $\overline{B}$ over k.

PROBATIO: Indeed, by definition, B has a uniformising element π such that

$$B^\dagger/\pi B^\dagger = \overline{B}.$$

If a presentation of $B^\dagger$ is given by $A^\dagger/(f_1, \ldots, f_m)$, let $\mathfrak{I}$ be the ideal generated by the $n-d \times n-d$ minors of the Jacobian $\frac{\partial F}{\partial x}$. Comme $\overline{B}/k$ est régulier de dimension d, l'idéal engendré par π et $\mathfrak{I}$ est $B^\dagger$. Donc $\mathfrak{I}$ contient un élément de la forme $1 - \pi b$ avec $b \in B^\dagger$. Étant donné que la série $1 + \pi b + \pi^2 b^2 + \ldots$ converge dans $B^\dagger$, $1 \in \mathfrak{I}$. Alors A est engendré par I. Alors par le critérion de Jacobi, $\Omega^\bullet_{B^\dagger/W(k)} \otimes_{\mathbb{Z}} \mathbb{Q}$ est projecive de rang d sur $B^\dagger$. □

Questions: **Est-ce indépendent du choix du relèvement? La construction, est-elle fonctorielle?**

Les reponses ne sont pas évidentes, mais affirmatives d'après van der Put. Il a aussi démontré qu'on peut relever des application:

Theorema 3.3. *Let $\overline{C}/k$ be smooth and finitely generated, let $C^\dagger$ be a lift of $\overline{C}$ and let $f : \overline{B} \to \overline{C}$ be a morphism of k-algebras. Then there exists a $W(k)$-morphism $F : B^\dagger \to C^\dagger$ lifting f.*

This is done using Artin approximation.

PROBATIO: Par définition de relèvement (d'après van der Put), $B^\dagger/W(k)$ est plat, et similaire pour $C^\dagger$. Du fait que $\overline{B} = B^\dagger/\pi$ et $\overline{C} = C^\dagger/\pi$ sont lisses sur $k = W(k)/\pi$, on en déduit que $B^\dagger/\pi^n$ et $C^\dagger/\pi^n$ le sont sur $W(k)/\pi^n$.De la lissitude on obtient un système projective de $W(k)$-morphisme

$$f_n : B^\dagger/\pi^n \to C^\dagger/\pi^n$$

tel que $f_1 = f$. Cela résulte dans la limite en un morphisme

$$\hat{F} : \hat{B} \to \hat{C}.$$

Par une modification d'Approximation d'Artin, on en déduit l'existence d'un morphisme $F : B^\dagger \to C^\dagger$ comme dans l'énoncé. □

Note that the lifting of a map is not unique though. En particulier, il exist un relèvement de Frobenius.

References

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