# 1 Chow groups

## 1.1 Definition

Let $X$ be a scheme (separated, noetherian, finite dimensional excellent – any separated scheme of finite type over a field or over $\operatorname{Spec}\mathbb{Z}$ satisfies these hypotheses). Consider the direct sum

$$R(X) := \oplus_{\zeta\in X} k(\zeta)^*$$

of $K_1$-chains on $X$, where $k(\zeta)$ is the residue field at the point $\zeta$. For noetherian schemes there is a natural grading of this group by codimension

$$R^q(X) := \oplus_{\zeta\in X^{(q)}} k(\zeta)^*$$

the group of co.dimension $q$ $K_1$-chains. If $X$ is finite dimensional there is also a grading by dimension and if $X$ is catenary and equidimensional these gradings are equivalent. In general, codimension is better behaved.

We denote by $Z(X)$ the group of cycles of $X$. SInce the closed integral subschemes of $X$ are in one to on correspondence with the points of $X$, identifying an integral subscheme with its generic point, we have

$$Z(X) = \oplus_{\zeta\in X}\mathbb{Z}.$$

The natural map from the group of Cartier divisors to the group of Weil divisors induces for an integral scheme a homomorphism

$$\operatorname{div} : k(X)^* \to Z^1(X).$$

Therefore if $X$ is as in the beginning, we get for each integral subscheme $Z \subset X$ a homomorphism $\div : k(Z)^* \to Z^1(Z)$ and summing over the integral subschemes yields a homomorphism

$$\operatorname{div} : R(X) \to Z(X)$$

. A cycle in the image is said to be rationally equivalent to zero, and this defines an equivalence relation on the group of cycles.

**Definition 1.1.** if $X$ is a general scheme, we set the ungraded Chow group of $X$ to be

$$\operatorname{CH}(X) := \operatorname{Coker}(\operatorname{div}).$$

If $X$ satisfies the assumptions of the beginning, the homomorphism div is of pure dimension $-1$ with respect to the grading by (relative) dimension, which induces a grading of the Chow group by dimension, namely by setting $\operatorname{CH}_q(X)$ equal to the cokernel of

$$\operatorname{div} : R_{q+1}(X) \to Z_q(X).$$

The map div is not as well behaved with respect to codimension (unless $X$ is equidimensional), but it increases the codimension by at least one, so it makes sense to define a grading of the Chow group with respect to codimension by setting $\operatorname{CH}^p(X)$ to be equal to the cokernel of

$$\begin{aligned} \operatorname{div} : \oplus_{x\in X^{(p-1)}} k(x)^* &\to \oplus_{x\in X^{(p)}}\mathbb{Z} \\ f = \sum_x \{ f_x\} &\mapsto \sum_x \operatorname{div}(\{f_x\}). \end{aligned}$$

If $X$ is equidimensional, then the gradings are compatible.

We can sheafify the funtors mentioned here and get flasque sheaves:

$$\begin{aligned} \mathscr{R}^q_X : U &\mapsto R^q(U) \\ \mathscr{Z}^p_X : U &\mapsto Z^p(U) \end{aligned}$$

and the divisor homomorphism gives a morphism of sheaves

$$\mathrm{div} : \mathscr{R}_X^{q-1} \to \mathscr{Z}_X^q$$

as well as an isomorphism

$$\mathrm{CH}^q(X) \cong \mathbb{H}^1(X, \mathscr{R}_X^{q-1} \to \mathscr{Z}_X^q).$$

It also induces a map $\mathrm{Div}(X) \to \mathrm{CH}^1(X)$ that can be seen to factor through $\mathrm{Pic}(X)$ and vanishes on principal divisors. Finally we get a cap product structure on the $\mathrm{CH}(X)$.

## 1.2 Basic Properties

**Functoriality:** Let $f : X \to Y$ be a morphism of schemes. If $f$ is flat, there is a pull-back $f^* : Z^p(Y) \to Z^p(X)$ preserving codimension, for a closed integral subscheme $Z \subset X$

$$f^* : [Z] \mapsto [\mathscr{O}_X \otimes_{\mathscr{O}_Y} \mathscr{O}_Z]\,,$$

which extends by linearity.

On the other hand, if $f$ is proper, there is a push-forward; for a closed integral $k$-dimensional subscheme $Z$

$$f_*([Z]) = \begin{cases} [k(Z) : k(f(Z))]\,[f(Z)] & \text{if } \dim(f(Z)) = \dim(Z) \\ 0 & \text{if } \dim(f(Z)) < \dim(Z). \end{cases}$$

Both, push-forward and pull-back are compatible with rational equivalences and therefore induce maps on Chow groups. If $f : X \to Y$ is flat we have

$$f^* : \mathrm{CH}^p(Y) \to \mathrm{CH}^p(X)$$

and if $f : X \to Y$ is proper, we have

$$f_* : \mathrm{CH}_q(X) \to \mathrm{CH}_q(Y).$$

**Product structure:** Recall that two cycles are said to meet properly, if their supports intersect properly. If two prime cycles meet properly, to each irreducible component $W \subset Y \cap Z$ there is assigned an integer $\mu_W(Y, Z)$ called the intersection multiplicity and this gives an intersection product for cylces that intersect properly

$$[Y]\,.\,[Z] = \sum_W \mu_W(Y, Z)\,[W]\,.$$

The next step is Chow's moving lemma:

**Theorem 1.2.** *Suppose that $X$ is a smooth quasi-projective variety over a field $k$, and $Y$ and $Z$ are integral subschemes. Then the cycle $[Y]$ is rationally equivalent to a cycle $\eta$ which meets $[Z]$ properly.*

This induces a product on $\mathrm{CH}(X)$ as follows:

**Theorem 1.3.** *Let $X$ be a smooth quasi-projective variety over a field $k$.*

- *For two elements $\alpha, \beta \in \mathrm{CH}(X)$ represented by two cycles $\eta$ and $\zeta$ which meet properly. Then the class of $\eta.\zeta$ in $\mathrm{CH}^*(X)$ is independent of the choice of representatives and depends only on $\alpha$ and $\beta$.*
- *The product on $\mathrm{CH}(X)$ defined by this is associatitve and commutative.*
- *The assignement $X \mapsto \mathrm{CH}^*(X)$ is a contravariant functor from the category of quasi-projective smooth varieties to the category of commutative rings.*

**Coniveau filtration:** Let $X^{\geqslant i}$ the family of supports consisting of subsets of codimension at least $i$ and $X_{\leqslant i}$ the subsets of dimension at most $i$.

**Definition 1.4.** The filtration by codimension of supports (or coniveau filtration) is the decreasing filtration for $i \in \mathbb{N}_0$

$$F^i_{\text{cod}}(K_0(X)) := \text{Im}\left(K_0^{X^{\geqslant i}}(X) \to K_0(X)\right).$$

Similar for $G_0(X)$.

Write $\text{Gr}^\bullet_{\text{cod}}(K_0(X))$ respectively $\text{Gr}^\bullet_{\text{cod}}(G_0(X))$ for the associated graded groups. If $Y \subset X$ is a subscheme of a noetherian scheme, then $[\mathscr{O}_Y] \in F^p(G_0(X))$. Thus we have a map

$$Z^p(X) \to F^p_{\text{cod}}(G_0(X))$$

and this induces by dévissage a surjective map

$$Z^p(X) \to \text{Gr}^p_{\text{cod}}(G_0(X)).$$

**Theorem 1.5.** *For an arbitrary noetherian scheme, this factors through* $\text{CH}^p(X)$*.*

This filtration is conjecture to be multiplicative. Multiplicity can be proved after tensoring with $\mathbb{Q}$ by comparing it with the $\gamma$-filtration.

# 2 Rost's Axioms

## 2.1 Milnor $K$-theory

For a field F we let

$$T^*(F) = \mathbb{Z} \oplus F^* \oplus (F^* \otimes F^*) \oplus \cdots$$

be the tensor algebra over the $\mathbb{Z}$-module $F^*$. Let $I$ be the two-sided homogeneous ideal in $T(F^*)$ generated by elements $a \otimes (1-a)$ with $a, 1-a \in F*$. These are called Steinberg relations.

**Definition 2.1.** The Milnor $K$-groups of a field $F$ are defined to be the graded ring

$$K^M_*(F) = T^*(F)/I.$$

The residue class of an element $a_1 \otimes a_2 \otimes \cdots \otimes a_n$ is denoted $\{a_1, a_2, \ldots, a_n\}$. It is clear that $K_0(F) = \mathbb{Z}$ and that $K_1(F) = F$ . This is functorial for inclusions of fields. There are some obvious relations such as

- For $x \in K_n(F)$ and $y \in K_m(F)$: $xy = (-1)^{nm} yx$ .
- If $a \in F^*$: $\{a, -a\} = \{a, -1\}$.
- From this can be deduced that if $a_1, \ldots, a_n \in F^*$ and $a_1 + \cdots + a_n$ is either 0 or 1 we have $0 = \{a_1, \ldots, a_n\} \in K_n(F)$ .

For some fields $K^M_*$ can be written down explicitely.

For a discretely valued field $(F, \nu)$ with ring of integers $A$ and prime element $\pi$, there exists a unique group homomorphisms $\partial : K^M_n(F) \to K^M_{n-1}(A/\pi)$ such that for $u_i \in A^*$

$$\begin{aligned} \partial\{\pi, u_2, \ldots, u_n\} &= \{u_2, ..., u_n\} \\ \partial\{u_1, \ldots, u_n\} &= 0. \end{aligned}$$

The next proposition is one of the basic results in Milnor $K$-theory due to Milnor.

**Proposition 2.2.** *For a field $F$ the sequence*

$$0 \to K^M_n(F) \to K^M_n(F(t)) \xrightarrow{\partial} \otimes_\pi K^M_{n-1}(F[t]/\pi) \to 0$$

*is split exact. Here the sum is over all irreducible, monic $\pi \in F[t]$.*

## 2.2 Milnor $K$-theory and Chow groups

A relationship between Milnor $K$-theory and complexes computing the Chow groups is explained in [5]. Rost considers in this paper the more general structure of cycle modules which includes Milnor $K$-theory.

**Definition 2.3.** A (graded) cylce module is a covariant functor $M$ from teh category of fields to the category of $\mathbb{Z}$-graded abelian groups together with the following:

1. For every finite field extension $F \subset E$ a transfer map:
$$\mathrm{tr}_{E/F} : M(E) \to M(F)$$
of degree zero.

2. For a field $F$ together with a discrete valuation $\nu$ a boundary map:
$$\partial_\nu : M(F) \to M(k(\nu))$$
of degree $-1$.

3. For every field $F$ a pairing
$$F^* \times M(F) \to M(F)$$
of degree one, extending to a pairing
$$K_*^M(F) \times M(F) \to M(F)$$
making $M(F)$ a graded modules over the Milnor $K$-theory ring.

By the facts about Milnor $K$-theory mentioned above, it is obvious that Milnor $K$-theory is a cycle module. Incidentally the same holds true for Quillen $K$-theory.

**Definition 2.4.** Let $X$ be a variety over a field and $M$ a cycle module. For $q \in \mathbb{Z}_0$ we define a complex $C^*(X, M, q)$ called the cohomological cycle complex associated to the cycle module $M$ via

$$C^p(X, M, q) = \oplus_{x \in X^{(p)}} M_{q-p}(k(x))$$

with differentials $C^p(X, M, q) \to C^{p+1}(X, M, q)$ induced by the boundary maps of the cycle module $\partial_\nu : M_{q-p}(k(x)) \to M_{q-(p+1)}(k(\nu))$ for each discrete valuation which is trivial on the ground field. In a similar way one can define a homological cycle complex for $M$

$$C_p(X, M, q) = \oplus_{x \in X_{(p)}} M_{q-p}(k(x))$$

One can prove that the cohomological complex for Milnor $K$-theory is contravariant for flat maps, and the homological one is covariant with respect to proper maps (by Weil reciprocity). And a similar statement holds for Quillen $K$-theory.

We denote the homology/cohomology of these complexes by

$$\begin{aligned} A_p(X, M, q) &:= \mathrm{H}_p(C_*(X, M, q)) \\ A^p(X, M, q) &:= \mathrm{H}^p(C^*(X, M, q)). \end{aligned}$$

From what we said above, the groups $A_*(X, M, q)$ respectively $A^*(X, M, q)$ are covariant for proper respectively contravariant for flat morphisms, where $M$ is Milnor or Quillen $K$-theory.

Fix $p$ and consider $C_*(X, M, p)$ for $M = K_*^M$. We know that for a field $F$, $K_1(F) = F^*$, $K_0(F) = \mathbb{Z}$ and $K_{<0}(F) = 0$. Therefore

$$\begin{aligned} C_{p-1}(X, M, p) &= \oplus_{x \in X_{(p-1)}} K_1(k(x)) = \oplus_{x \in X_{(p-1)}} k(x)^* = R_{p-1}(X) \\ C_p(X, M, p) &= \oplus_{x \in X_{(p)}} K_0(k(x)) = \oplus_{x \in X_{(p)}} \mathbb{Z} = Z_p(X) \\ C_{>p}(X, M, p) &= 0 \end{aligned}$$

**Proposition 2.5.** *If $M$ is Milnor (or Quillen) $K$-theory we have*

$$\begin{aligned} A_p(X, M, -p) &\cong \mathrm{CH}_p(X) \\ A^p(X, M, p) &\cong \mathrm{CH}^p(X) \end{aligned}$$

PROOF: This follows exactly from the definitions. □

The following statements are proved by Rost in [5].

**Theorem 2.6.** *Let $M$ be a cycle module.*

1. *The cohomology groups $A^*(X, M, *)$ are homotopy invariant. More precisely, For a flat morphism $\pi : E \to X$ with affine spaces as fibers (an affine fibration), the pull-back morphism $\pi^* : A^*(X, M, *) \to A^*(E, M, *)$ is an isomorphism.*

2. *If $f : X \to S$ is flat where $S$ is (the spectrum of) a Dedekind ring $\Lambda$, and $t$ a regular element of $\Lambda$, there is a specialisation map*

$$\sigma_t : A^*(X_t, M, *) \to A^*(X_0, M, *)$$

*which preserves the bigrading. (Recall: $X_t = X \times_S \operatorname{Spec}(\Lambda\left[\frac{1}{t}\right])$ and $X_0 = X \times_S \operatorname{Spec}(\Lambda/(t))$.)*

3. *If in addition $M$ has a ring structure, and $f : Y \to X$ is a regular immersion, then there is a Gysin homomorphism*

$$f^* : A^*(X, M, *) \to A^*(Y, M, *).$$

*This Gysin map is compatible with flat pull-backs in the following sense:*

   (a) *If $f : Z \to Y$ is flat and $p : X \to Y$ a regular immersion, consider the diagram*

$$\begin{array}{ccc} X \times_Y Z & \xrightarrow{i_z} & Z \\ {\scriptstyle p_x}\downarrow & & \downarrow{\scriptstyle p}\ \textit{regular immersion} \\ X & \xrightarrow[\textit{flat}]{i} & Y \end{array}$$

   *then*

$$p_X^* i^* = i_Z^* p^*.$$

   (b) *If $p : X \to Y$ is flat and $i : Y \to X$ a section of $p$ which is a regular immersion, then*

$$i^* p^* = \mathrm{id}_Y^* .$$

4. *We assume again that $M$ has a ring structure. If $X$ is a smooth variety over a field, then $A^*(X, M, *)$ has a product structure.*

**Corollary 2.7.** *For all $p, q \geqslant 0$ the assignement*

$$X \mapsto A^p(X, M, q)$$

*is a contravariant functor from the category of smooth varieties over $k$ to abelian groups.*

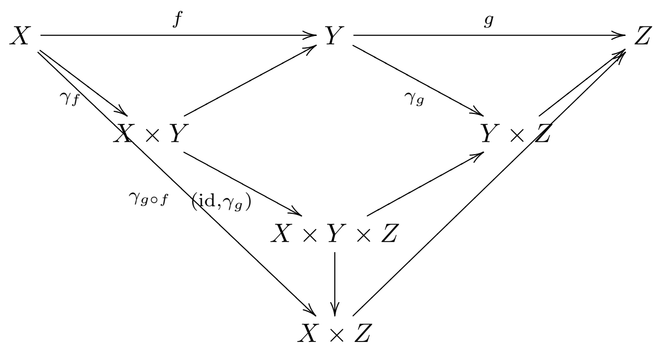
Indeed, let $f : X \to Y$ be a morphism of smooth $k$-varieties. This can be factored as

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle \gamma_f}\searrow \quad \nearrow{\scriptstyle p} & \\ & X \times_k Y & \end{array}$$

where the projection $p$ is flat and $\gamma_f$ – the graph of $f$ – is a regular immersion. Thus we can define for $A^p(\cdot, M, q)$

$$f^* = \gamma_f^* \circ p^* : A^p(Y, M, q) \to A^p(X, M, q).$$

To show that this is compatible with composition, note that for morphisms of smooth varieties $f : X \to Y$ and $g : Y \to Z$ we have a commutative diagram of the form




This shows that $p_g \cdot \gamma_g \circ p_f \cdot \gamma_f = g \circ f = p_{gf} \cdot \gamma_{gf}$. One has to use part (3) of the theorem to show that this is compatible with pull-backs.

**Definition 2.8.** On the big Zariski site of regular varieties we define the sheaf associated to a cycle complex $M$ by

$$\mathscr{M}_q : X \mapsto A^0(X, M, q) = \mathrm{H}^0(C^*(X, M, q)) = \mathrm{Ker}(\oplus_{x \in X^{(0)}} M_q(k(x)) \to \oplus_{x \in X^{(1)}} M_{q-1}(k(x)).$$

In particular, this defines the Milnor $K$-sheaf.

By a variation of the proofs of the Gersten conjecture by Quillen and Gabber, one can show the following [5]

**Theorem 2.9.** *If $X$ is the spectrum of a regular semi-local ring, which is a localisation of an algebra of finite type over $k$, then $\forall p \in \mathbb{Z}$ the complex $C^*(X, M, p)$ only has cohomology in degree* 0*.*

Our updated Terms of Use will become effective on May 25, 2012. Find out more. Sheaf cohomology From this follows via a spectral sequence argument:

**Corollary 2.10.** *If $X$ is a regular variety over $k$, then $\mathrm{H}^p(C^*(X, M, q)) \cong \mathrm{H}^p(X, \mathscr{M}_p)$.*

With Proposition 2.5 we can now compute the Chow group:

**Corollary 2.11.** *If $X$ is a regular variety over $k$, then $\mathrm{CH}^p(X) \cong \mathrm{H}^p(X, \mathscr{M}_p)$.*

### 2.3 Chern classes into Milnor $K$-sheaves

To construct Chern classes we can use the methods of [1]. We have that $\mathscr{K}_1(X) = \mathscr{O}_X^*$ and since $M_*$ is by definition a $K_*^M$-module, there are products for $p$ and $q$ varying

$$\mathrm{H}^1(X, \mathscr{O}_X^*) \otimes A^p(X, M, q) \to A^{p+1}(X, M, q+1).$$

We can then prove a projective bundle formula.

**Theorem 2.12.** *Let $M_*$ be a cycle module, $X$ a $k$-variety, and $\pi : E \to X$ a vector bundle of constant rank $n$. Then there is an isomorphism*

$$A^p(\mathbb{P}(E), M, q) \cong \oplus_{i=0}^{n-1} A^{p-i}(X, M, q-i)\xi^i$$

*where $\xi \in \mathrm{H}^1(\mathbb{P}(E), \mathscr{O}_{\mathbb{P}(E)}^*)$ is as usual the class of $\mathscr{O}_{\mathbb{P}(E)}(1)$.*

This is proved by a spectral sequence argument using the Gysin homomorphism [2]. Following the argumentation of [1, Theorem 2.2] we obtain atheory of Chern classes in the following sense:

**Theorem 2.13.** *There is a theory of Chern classes for higher algebraic $K$-theory on the category of regular varieties over $k$, with values in Zariski cohomology with coefficients in Milnor $K$-sheaves*

$$c_n : K_p(X) \to \mathrm{H}^{n-p}(X, \mathscr{K}_n^M).$$

**Conclusion: We obtained indeed a theory of Chern classes into Milnor $K$-sheaves satisfying the axioms given by Gillet [1]. There seems to be no condition on the base field $k$.**

## References

[1] Gillet H.: *Riemann-Roch Theorems for Higher Algebraic $K$-Theory.* Advances in Mathematics 40, 203-289, (1981).

[2] Gillet H.: *$K$-Theory and Intersection Theory.* In Handbook of $K$-Theory, http://www.math.uiuc.edu/K-theory/handbook/, Springer-Verlag Berlin Heidelberg, (2005).

[3] Kerz M.: *Milnor $K$-theory of local rings.* Thesis, Regensburg, (2008).

[4] Milnor J.: *Algebraic $K$-theory and quadratic forms.* Invent. Math. 9, 318-344, (1969/1970).

[5] Rost M.: *Chow Groups with Coefficients.* Doc.Math.,1:No. 16, 319-393,(1996).