1 General formalism

Let $\mathcal{C}$ be an exact category (in the sense of Quillen): a full additive subcategory of an abelian category that is closed under extensions) endowed with two maps called degree and rank on the isomorphism classes of object in $\mathcal{C}$

$$\deg : \mathrm{Ob}\,\mathcal{C} \to \mathbb{R} \qquad \text{and} \qquad \mathrm{rank} : \mathrm{Ob}\,\mathcal{C} \to \mathbb{N}$$

additive on the exacte sequences of $\mathcal{C}$. We assume further that there is an abelian category $\mathcal{A}$ and a "generic fiber functor"

$$F : \mathcal{C} \to \mathcal{A}$$

such that

1. F is exact and faithful,
2. it induces a bijection

   $$F : \{\text{strict subobjects of } X\} \xrightarrow{\sim} \{\text{subobjects of } F(X)\},$$

   where strict means that they fit into an exact sequence.
3. the rank map on $\mathcal{C}$ comes from composition of an additive (rank) map on $\mathcal{A}$, $\mathrm{rank} : \mathcal{A} \to \mathbb{N}$ verifying

   $$\mathrm{rank}(X) = 0 \qquad \Leftrightarrow \qquad X = 0.$$

4. if $u : X \to X'$ is a morphism in $\mathcal{C}$ such that the induced $F(u)$ is an isomorphism, then

   $$\deg(X) \leq \deg(X'),$$

   equality if and ony if u was an isomorphism in the first place.

Especially the last condition is crucial in the definition.

Think of the inverse of the bijection induced by F as a sort of schemetheoretic closure operator. This will become clearer in the examples we discuss.

The notion of an exact category was first introduced by Quillen in order to describe in a non-abelian category the properties of short exact sequences without asking that morphisms actually have kernels and cokernels. Every abelian category is of course exact. With the additional conditions we impose on our category we can get even closer to an abelian category, it is a **quasi-abelian category** in the sense of Yves André. And infect, in this case every morphism disposes of a kernel and cokernel. This comes of course from the nice properties of F, basically we want that "kernel and image are stable under F".

More precisely: let $u : X \to Y$ in $\mathcal{C}$. Then $\mathrm{Ker}\, u$is the unique strict subobject $X' \subset X$ such that

$$F(X') = \mathrm{Ker}(F(u)).$$

Furthermore, $\mathrm{Im}\, u$ is the unique stict subobject $X" \subset X$ such that

$$F(X") = \mathrm{Im}(F(u)).$$

This defines the cokernel to be $coker u = X/\,\mathrm{Im}\, u$.

Remark 1.1. However, the category fails to be abelian in general: triviality of kernel and cokernel of a morphism doesn't necessarily imply that it is an isomorphism. Note that the induced morphism

$$X/\,\mathrm{Ker}\, u \to \mathrm{Im}\, u$$

is not in general an isomorphism, but it becomes one "in the generic fiber", i.e. once we apply the functor F on it. Looking at the last an crucial condition we mentioned in our definition, it is an isomorphism if and only if $\deg(X/\,\mathrm{Ker}\, u) = \deg(\mathrm{Im}\, u)$.

For $0 \neq X \in \mathcal{C}$ we set the slope

$$\mu(X) = \frac{\deg(X)}{\mathrm{rank}(X)} \in \mathbb{R}\,.$$

Definition 1.2. An object $0 \neq X \in \mathcal{C}$ is said to be semi-stable if for each non-zero strict sub-object $X' \subset X$

$$\mu(X') \leq \mu(X).$$

We have the following theorem due to Harder-Narasimhan:

Theorem 1.3. *Under the previous conditions on $\mathcal{C}$ every object $X \in \mathcal{C}$ has a unique filtration*

$$0 = X_0 \subsetneq X_1 \subsetneq \cdots \subsetneq X_r = X$$

in $\mathcal{C}$ such that

- *X_i/X_{i-1} is semi-stable,*
- *the sequence of slopes $(\mu(X_i/X_{i-1}))_{1\leq i\leq r}$ is strictly decreasing.*

For X as in this statement we denote by $\mathrm{HN}(X)$ the unique concave polygon with origin in $(0,0)$ with slopes $\mu(X_i/X_{i-1})$ and mutliplicities $\mathrm{rank}(X_i/X_{i-1})$ (i.e. thevertices have x-coordinate the rank and y-coordinate the degree). It is the concave hull of the points $(\deg(X'), \mathrm{rank}(X'))$ as X' runs through the subobjects of X. (Question: Shouldn't the coordinates be reversed here??) (Or in other words the convex hull of the points $(\mathrm{rank}(X'), \deg(X'))$.)

Theorem 1.4. *If $X' \subset X$ is a strict subobject, then the point $(\deg(X'), \mathrm{rank}(X'))$ lies under the polygone.*

For $\lambda \in \mathbb{R}$ we consider the following categories:

- $\mathcal{C}^{\leq\lambda} \subset \mathcal{C}$ the full subcategory of objects such that the slopes of their HN is at most λ. So $X \in \mathcal{C}^{\leq\lambda}$ iff for all strict subobjects $Y \neq 0$, $\mu(Y) \leq \lambda$.
- $\mathcal{C}_{\geq\lambda} \subset \mathcal{C}$ the full subcategory of objects such that the slopes of their HN is at least λ. So $X \in \mathcal{C}_{\geq\lambda}$ iff for all strict epimorphisms $X \twoheadrightarrow Y \neq 0$ we have $\mu(Y) \geq \lambda$.
- $\mathcal{C}^{st}_{\lambda} = \mathcal{C}^{\leq\lambda} \cap \mathcal{C}_{\geq\lambda}$ the full subcategory of semi-stable objects of slope λ inclucing the zero-object.

Theorem 1.5. *We have the following statments:*

1. *For $\lambda \in \mathbb{R}$ the categories $\mathcal{C}_{\geq\lambda}$ and $\mathcal{C}^{\leq\lambda}$ are exact and stable under ectension in $\mathcal{C}$.*
2. *For $\lambda > \mu$ $\mathrm{Hom}(\mathcal{C}_{\geq\lambda}, \mathcal{C}^{\leq\mu}) = 0$. In particular, if X is semistable of slope λ and Y is semistable of slope $\mu < \lambda$, $\mathrm{Hom}(X, Y) = 0$.*
3. *For all $\lambda \in \mathbb{R}$, $\mathcal{C}^{st}_{\lambda}$ is an abelian category stable under extension in $\mathcal{C}$.*

Consequently, the Harder-Narasimhan filtrations provide a canonical dévissage of the exact category $\mathcal{C}$ by the family of abelian categories $(\mathcal{C}^{st}_{\lambda})_{\lambda}$. On step further:

Definition 1.6. An object $X \in \mathcal{C}$ is stable if for all strict nontrivial subobjects $X' \subset X$, $\mu(X') < \mu(X)$.

And we have the following

Proposition 1.7. *For $\lambda \in \mathbb{R}$, all objects of $\mathcal{C}^{st}_{\lambda}$ are of finite length. The simple objects of $\mathcal{C}^{st}_{\lambda}$ are the stable objects of slope λ.*

2 Examples

2.1 Vector bundles

X a complete curve and $\mathcal{C}$ the category of locally free $\mathcal{O}_X$-modules of finite rank over X. As we have seen, we have an additive degree and an additive rank map over X. Let moreover $\mathcal{A}$ be the abelian category of finite dimensional $F(X)$-vector spaces ($F(X) = \mathcal{O}_{X,\eta}$). There is an obvious generic fiber functor $\mathcal{C} \to \mathcal{A}$. Now one has to verify that the conditions for the existence of a Harder Narasimhan filtration are satisfied. For example for a morphism $u : \mathcal{E} \to \mathcal{E}'$ which is an isomorphism on the generic fiber, then

$$\deg(\mathcal{E}') = \deg(\mathcal{E}) + \deg(\mathcal{E}'/u(\mathcal{E}))$$

where the degree of the coherent torsion sheaf $\mathcal{F} = \mathcal{E}'/u(\mathcal{E})$ is

$$\deg(\mathcal{F}) = \sum_{x\in|X|} \deg(x). \operatorname{long}_{\mathcal{O}_{X,x}}(\mathcal{F}_x).$$

2.2 Filtered vector spaces

Let L/K be a field extension, $\mathcal{V}ect\mathcal{F}il_{L/K}$ the exact categorie of pairs $(V, \operatorname{Fil}^\bullet V_L)$, where V is a finite dimensional K-vector space together with a filtration of $V \otimes_K L$ such that $\operatorname{Fil}^i V_L = 0$ for $i \gg 0$ and $\operatorname{Fil}^i V_L = V_L$ for $i \ll 0$. Define the rank and the degree as follows:

$$\begin{aligned} \operatorname{rank}(V, \operatorname{Fil}^\bullet V_L) &= \dim_K V, \\ \deg(V, \operatorname{Fil}^\bullet V_L) &= \sum_{i\in\mathbb{Z}} i. \dim_L \operatorname{gr}^i V_L \end{aligned}$$

Let $\mathcal{V}ect_K$ be the category of finite dimensional K- vector spaces. The forgetful functor

$$\begin{aligned} F : \mathcal{V}ect\mathcal{F}il_{L/K} &\to \mathcal{V}ect_K \\ (V, \operatorname{Fil}^\bullet V_L) &\mapsto V \end{aligned}$$

has the required properties. Thus we have a Harder-Narasimhan filtration in this category. The previous theorem tells us, that in this case every morphism between semi-stable objects of the same slope is compatible with the filtrations. (Exercise: why?)

2.3 Isocristals

Let k be a perfect field of characteristic $p > 0$, $K_0 = W(k)[\frac{1}{p}]$, σ the Frobenius on K_0. We consider the abelian, $\mathbb{Q}_p$-linear category $\phi - \mathcal{M}od_{K_0}$ of k-isocristals – consisting of pairs (D, ϕ) where D is a finite-dimensional K_0-vector space and $\phi : D \xrightarrow{\sim} D$ a σ-linear isomorphism. There are two additive functors, the hight and the end point of the Newton polygon

$$\begin{aligned} \operatorname{ht} &: \phi - \mathcal{M}od_{K_0} \to \mathbb{N} \\ t_N &: \phi - \mathcal{M}od_{K_0} \to \mathbb{Z} \end{aligned}$$

where $\operatorname{ht}(D, \phi) = \dim_{K_O} D$ and $t_N(D, \phi) = d$ if $\det(D, \phi) = K_0.e$ with $\phi(e) = a.e$ and $\nu_p(a) = d$. The hypothesis are satisfied if we take t_N for the degree map, and ht for the rank map. Note that the category is already abelian, so we don't have to take an injection. One can verify that the obtained Harder-Narasimhan filtration is fact the Dieudonné-Manin filtration and the Harder-Narasimhan polygon (concave) can be obtained from the Newton polygon (convex) by reversing the order of the slopes.

2.4 Filtered ϕ-modules

Combine here the previous two examples. Notation like in previous example and K/K_0 a field extension. The category $\phi - \mathcal{M}od\mathcal{F}il_{K/K_0}$ of triples $(D, \phi, \mathrm{Fil}^\bullet D_K)$ where $(D, \phi) \in \mathcal{M}od^{\phi}_{K_0}$ and $\mathrm{Fil}^\bullet D_K$ is a decreasing filtration of $D \otimes_{K_0} K$ with same conditions as before. This is an exact category, where the exact sequences are the sequences of isocristals that are strictly compatible with filtrations. Consider the additive end point map of the Hodge polygon

$$\begin{array}{rcl} t_h : \phi - \mathcal{M}od\mathcal{F}il_{K/K_0} & \to & \mathcal{V}ect\mathcal{F}il_{K/K_0} \xrightarrow{\deg} \mathbb{Z} \\ (D, \phi, \mathrm{Fil}^\bullet D_K) & \mapsto & (D, \mathrm{Fil}^\bullet D_K) \end{array}$$

As rank mal we take the map $(D, \phi, \mathrm{Fil}^\bullet D_K) \mapsto \mathrm{ht}(D, \phi)$ and as degree map the function $t_H - t_N$. The forgetful functor

$$\phi - \mathcal{M}od\mathcal{F}il_{K/K_0} \to \mathcal{M}od^{\phi}_{K_0}$$

satisfies the properties for a generic fiber functor and therefore we obtain a Harder-Narasimhan filtration.The abelian category of semi-stable objects of slope 0 is the category of weakly admissible filtered ϕ-modules in the sense of Fontaine.

2.5 ϕ-modules over the Robba ring

Maybe later.

2.6 Breuil-Kisin modules

2.7 Finite flat group schemes

References

[1] Chen, H.: *Harder-Narasimhan Categories.* Journal of pure and applied algebra 214, no. 2, 187-200, (2010).

[2] Fargues, L.; Fontaine, J.-M.: *Courbes et fibrés vectoriels en théorie de Hodge p-adique.* http://www-irma.u-strasbg.fr/~fargues/Prepublications.html (Courbe), (2011).

[3] *The Harder-Narasimhan Filtration.* Based on talks by Markus Reineke at the ICRA 12 conference in Torun, August 2007, www.math.uni-bielefeld.de/~sek/select/fahrhn.pdf, (2007).