LE NOYAU DE LA MONODROMIE EN COHOMOLOGIE p-ADIQUE – UNE APPROCHE RIGIDE ANALYTIQUE

par

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Résumé. — La cohomologie rigide (non-logarithmique) a des bonnes propriétés mêmes pour les variétés non-lisse, mais du point de vue des fonctions L p-adique elle n'est probablement pas la bonne cohomologie p-adique pour les variétés non-lisses – par exemple dans le cas semistable c'est plutôt la cohomologie de Hyodo–Kato qu'il faut considérer. Une conjecture de Flach–Morin cherche à déterminer de façon précise la différence entre ces deux théories de cohomologie p-adiques. Je vais présenter une approche à cette conjecture fondée sur des méthodes rigides analytique, introduire quelques constructions clés en cohomologie rigide logarithmique et expliquer un cas particuliers. (Travail en cours en commun avec Kazuki Yamada, Keio University.)

***Abstract* (The kernel of the monodromy in p-adic cohomology – a rigid analytic approach)**

(Non-logarithmic) rigid cohomology has good properties even for non-smooth varieties, but from the standpoint of p-adic L-functions, it is probably not the right p-adic cohomology to consider for non-smooth schemes – for example in the semistable case, one should use the Hyodo–Kato cohomology. A conjecture by Flach–Morin predicts the exact difference between these two p-adic cohomology theories. I will present an approach to this conjecture which relies on rigid analytic methods, introduce key constructions of log rigid cohomology and explain a particular case. (Work in progress joint with Kazuki Yamada.)

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It is a particular honour for me to speak here today, as the research of p-adic cohomology theories is a historically strong area here.

Today I want to talk about some constructions concerning rigid cohomology, more precisely log rigid cohomology, obtained together with my collaborator Kazuki Yamada from Keio University, and put this into the context of other p-adic cohomologies with the goal to draw a bigger picture.

The Weil conjectures can be seen as a starting point for the study of p-adic cohomology theories. Weil has suggested to use a suitable cohomology theory to solve these conjectures for proper and smooth varieties over a field k of characteristic p. For $\ell \neq p$, this has long been solved by Grothendieck's school

Mots clefs. — Triangle de localisation, géométrie logarithmique, cohomologie rigide, cohomologie cristalline.

using ℓ-adic cohomology. One motivation for the research of p-adic cohomology theories is the desire to fill the gap for $\ell = p$.

A first candidate was Berthelot's crystalline cohomology $H^*_{\mathrm{cris}}(X/W(k))$ [**1**] after a suggestion by Grothendieck. A drawback of crystalline cohomology is that it works well only for proper and smooth schemes – for singular or non-proper schemes, the crystalline cohomology groups are not necessarily finitely generated over $W(k)$. In this case rigid cohomology $H^i_{\mathrm{rig}}(X/K)$ introduced by Berthelot [**2**], [**3**] has become an important tool.

However, from the point of view of the study of p-adic L-functions, it doesn't seem to be the correct cohomology to consider for non-smooth schemes. One is lead to consider certain logarithmic cohomology theories instead, in particular the so called Hyodo–Kato theory [**8**].

Thus an important question to ask is, what is the difference of these two theories.

Question 0.1. — *Can we describe the difference and relation between rigid cohomology and Hyodo–Kato cohomology in mathematical term ?*

This is one of the motivating questions of our research.

Notation 0.2. — I will use the following notation :

V – complete discrete valuation ring of mixed characteristic $(0, p)$;
$\mathfrak{m}$ – its maximal ideal ;
K – its fraction field ;
$\overline{K}$ – an algebraic closure of K ;
G_K – the absolute Galois group of K ;
k – its residue field, which is perfect ;
$W(k)$ – the ring of Witt vectors of k ;
F – the fraction field of $W(k)$.

For a scheme X/V we denote by

X_n – for $n \in \mathbb{N}$, the reduction of X modulo p^n ;
X_0 – its special fibre ;
X_K – its generic fibre.

1. A conjecture by Flach and Morin

We consider the following situation :

X – a flat projective regular V-scheme of relative dimension d.

In this situation Flach and Morin in [**7**] conjecture the following :

Conjecture 1.1. — *There is an exact triangle in the derived category of φ-modules*

(1)
$$R\Gamma_{\mathrm{rig}}(X_0/F) \xrightarrow{\mathrm{sp}} \left[R\Gamma^B_{\mathrm{HK,h}}(X_{\overline{K}})^{G_K} \xrightarrow{N} R\Gamma^B_{\mathrm{HK,h}}(X_{\overline{K}})(-1)^{G_K} \right] \xrightarrow{\mathrm{sp}'} R\Gamma_{\mathrm{rig}}(X_0/F)^*(-d-1)[-2d-1] \to$$

where sp *is the specialisation map and* sp$'$ *is the composition of the Poincaré duality morphism*

$$R\Gamma^B_{\mathrm{HK,h}}(X_{\overline{K}})(-1) \cong R\Gamma^B_{\mathrm{HK,h}}(X_{\overline{K}})^*(-d-1)[-2d] \to$$

and the dual sp*.

This triangle can be interpreted as a localisation triangle.

Scholie 1.2. — Consider the following situation :

S	–	the unit disc;
$f : X \to S$	–	a proper flat analytic Kähler variety of relative dimension d;
$X_t = f^{-1}(t)$	–	mooth fibres for $t \in S^\times = S\backslash\{0\}$;
$Y = f^{-1}(0)$	–	the special fibre which is a (strict) normal crossings divisor in X;
$X^\times = X\backslash Y$	–	the complement of Y.

In the following : $H^n(-)$ will refer to the Betti cohomology.

We recall several facts :

(i) There is a natural morphism $r : X \to Y$ (called retraction) which induces an isomorphism on Betti (co)homology.

(ii) There is a long exact sequence of rational mixed Hodge structures

$$H^n(X^\times) \to H^n(X_\infty) \xrightarrow{N} H^n(X_\infty)(-1) \to$$

where $H^n(X_\infty)$ is the limit mixed Hodge structure of the generic fibre [**11**]. It is called the monodromy sequence.

(iii) There is a natural exact sequence

$$H^n_Y(X) \to H^n(X) \to H^n(X^\times) \to$$

called the localisation triangle. It can be deduced using the six functor formalism.

(iv) By Poincaré duality we have the following isomorphisms

$$\begin{array}{rcl} H^n_Y(X) & \cong & H^{2d+2-n}(Y)^*, \\ H^n(X^\times) & \cong & H_c^{2d+2-n}(X^\times)^*. \end{array}$$

Let $R\Gamma(X_\infty)$ be a complex computing the limit mixed Hodge structure and assume that there is a $\nu : R\Gamma(X_\infty) \to R\Gamma(X_\infty)(-1)$ inducing the monodromy N on cohomology, then

$$H^n(X^\times) \cong H^n\left(\left]R\Gamma(X_\infty) \xrightarrow{\nu} R\Gamma(X_\infty)(-1)\right]\right)$$

and the localisation sequence is induced by

$$R\Gamma_Y(X) \to R\Gamma(X) \to \left[R\Gamma(X_\infty) \xrightarrow{\nu} R\Gamma(X_\infty)(-1)\right] \to$$

On the other hand, using Poincaré duality the monodromy sequence can be interpreted as the local invariant cycle theorem

$$H^n(Y) \xrightarrow{\mathrm{sp}} H^n(X_\infty) \xrightarrow{N} H^n(X_\infty)(-1)$$

where sp is the composition $H^n(Y) \cong H^n(X) \to H^n(X^\times) \to H^n(X_\infty)$.

If one connects the monodromy exact sequence and the localisation exact sequence, one obtains a 4-term exact sequence

$$H^n(X) \to H^n(X_\infty) \xrightarrow{N} H^n(X_\infty)(-1) \to H^{n+2}_Y(X) \to$$

Identifying $H^n(X) \cong H^n(Y)$ by retraction and using Poincaré duality, we obtain the sequence

$$H^n(Y) \to H^n X_\infty) \xrightarrow{N} H^n(X_\infty)(-1) \to H_{2d-2}(Y) \to$$

which we call the Clemens–Schmid sequence [**10**].

Thus the triangle (1) can be interpreted as a localisation triangle by the following identifications :

$$\begin{aligned} R\Gamma_{\mathrm{rig}}(X_0/F) &\leftrightarrow H^n(Y), \text{ the cohomology of the special fibre;} \\ \left[R\Gamma^B_{\mathrm{HK,h}}(X_{\overline{K}})^{G_K} \xrightarrow{N} R\Gamma^B_{\mathrm{HK,h}}(X_{\overline{K}})(-1)^{G_K}\right] &\leftrightarrow H^n(X^\times), \text{ the cohomology of the open complement;} \\ R\Gamma_{\mathrm{rig}}(X_0/F)^*(-d-1)[-2d-1] &\leftrightarrow H^{n+1}_Y(X), \text{ the dual of the cohomology of the special fibre.} \end{aligned}$$

Let now Y/k be a proper strictly semistable scheme. In this case, the conjecture by Flach–Morin suggests that we should have a triangle of the form

$$R\Gamma_{\mathrm{rig}}(Y/F) \to \left[R\Gamma^{\mathrm{cris}}_{\mathrm{HK}}(Y) \xrightarrow{N} R\Gamma^{\mathrm{cris}}_{\mathrm{HK}}(Y)(-1)\right] \to R\Gamma^*_{\mathrm{rig}}(Y/F)(-d-1)[-2d-1] \to$$

And indeed, this can be deduced from results due to Chiarellotto and Tsuzuki [**6**] in the following case : $f : X \to C$ is a proper flat morphism over k where X is a smooth variety of dimension $d+1$ and C is a smooth curve such that for a k-rational point $s \in C$ the fibre $Y := X_s$ is a normal crossing divisor in X. They even obtain a full Clemens–Schmid exact sequence.

In order to generalise this result we have to study the building blocks that appear in Flach–Morin's conjecture, in particular the Hyodo–Kato theory.

2. Hyodo–Kato theory

Let us first recall what we mean if we say Hyodo–Kato theory.

Definition 2.1. — By a Hyodo–Kato theory for K-varieties or V-schemes X, we mean

(i) a cohomology theory $H^*_{\mathrm{HK}}(X)$ in finite dimensional F-vector spaces;

(ii) a bijective Frobenius-linear operator $\varphi : H^*_{\mathrm{HK}}(X) \to H^*_{\mathrm{HK}}(X)$, called **Frobenius**.

(iii) a nilpotent operator $N : H^*_{\mathrm{HK}}(X) \to H^*_{\mathrm{HK}}(X)$ such that $N\varphi = p\varphi N$, called the **monodromy**.

(iv) a functorial morphism $\Psi : H^*_{\mathrm{HK}}(X) \to H^*_{\mathrm{dR}}(X_K)$, which is an isomorphism after $\otimes K$, called the **Hyodo–Kato morphism**.

There are several constructions :

(i) Hyodo–Kato's original construction based on log crystalline cohomology. The Hyodo–Kato morphism Ψ^{HK}_π depends on the choice of a uniformiser π of V.

(ii) Beilinson's representation of the Hyodo–Kato complex with a Hyodo–Kato morphism Ψ^B **independent** of the choice of a uniformiser.

(iii) Große-Klönne's rigid analytic construction, using dagger spaces. The Hyodo–Kato map Ψ^{GK}_π depends on the choice of a uniformiser and is a zigzag through rather complicated intermediate objects.

Our goal was to obtain a Hyodo–Kato theory that **lends itself for computations** and is **indepenent of the choice of a uniformiser**.

Construction 2.2. — *(Ertl–Yamada) Let X/V be semistable. Using weak formal schemes and dagger spaces, we obtain*

– *a new presentation of the Hyodo–Kato cohomology*

$$R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(X)$$

together with a Frobenius φ and monodromy operator N ;

– *a natural functorial morphism*

$$\Psi : R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(X) \to R\Gamma_{\mathrm{dR}}(X_K)$$

which is a quasi-isomorphism after $\otimes K$. It has the following advantages :

– *It is **not** a zigzag.*

– *It is independent of the choice of a uniformiser.*

– *It is suitable for computations*

I will explain the construction locally. We will use the following log schemes

$$\begin{array}{rcl} k^0 & - & (\operatorname{Spec} k, 1 \mapsto 0) \\ W(k)^0 & - & (\operatorname{Spec} W(k), 1 \mapsto 0) \\ W(k)^{\varnothing} & - & (\operatorname{Spec} W(k), \mathrm{triv}) \\ V^{\sharp} & - & (\operatorname{Spec} V, \mathrm{can}) \\ \mathcal{S} & - & (\operatorname{Spwf} W(k)[\![s]\!], 1 \mapsto s) \end{array}$$

Now consider the situation

$$\begin{array}{rcl} Y & - & \text{semistable over } k^0\,; \\ \mathcal{Z} & - & \text{a lift to } \mathcal{S} \Rightarrow \text{log smooth over } W(k)^{\varnothing}\,; \\ \mathcal{X} & - & \mathcal{Z} \times V^{\sharp}; \\ \mathcal{Y} & - & \mathcal{Z} \times W(k)^0; \\ \mathfrak{Z}, \mathfrak{X}, \mathfrak{Y} & - & \text{the associated dagger spaces.} \end{array}$$

Now we can compute different log rigid cohomologies

$$\begin{array}{rcl} \omega^\bullet_{\mathcal{Z}/W(k)^{\varnothing},\mathbb{Q}} & - & \text{computes the ``absolute'' rigid cohomolog } R\Gamma_{\mathrm{rig}}\,(Y/W(k)^{\varnothing})\,; \\ \omega^\bullet_{\mathcal{Z}^{\varnothing}/W(k)^{\varnothing},\mathbb{Q}} & - & \text{computes the non-logaritmic rigid cohomology } R\Gamma_{\mathrm{rig}}\,(Y^{\varnothing}/W(k)^{\varnothing}) = R\Gamma_{\mathrm{rig}}\,(Y/F)\,; \\ \omega^\bullet_{\mathcal{X}/V^{\sharp},\mathbb{Q}} & - & \text{computes } R\Gamma_{\mathrm{rig}}\,(Y/V^{\sharp})\,; \\ \omega^\bullet_{\mathcal{Y}/W(k)^0,\mathbb{Q}} & - & \text{computes } R\Gamma_{\mathrm{rig}}\,(Y/W(k)^0)\,; \text{ should give Hyodo–Kato theory}\,; \\ \widetilde{\omega}^\bullet_{\mathcal{Y},\mathbb{Q}} & - & \text{the auxiliary complex } \omega^\bullet_{\mathcal{Z}/W(k)^{\varnothing},\mathbb{Q}} \otimes_{\mathcal{O}_{\mathfrak{Z}}} \mathcal{O}_{\mathfrak{Y}}\,; \end{array}$$

We now consider so called Kim–Hain complexes :

$$\omega^\bullet_{\mathcal{Z}/W(k)^{\varnothing},\mathbb{Q}}[u] \quad \text{and} \quad \widetilde{\omega}^\bullet_{\mathcal{Y},\mathbb{Q}}[u]$$

with $u^{[i]}$ of degree 0, such that $du^{[i+1]} = d\log s \cdot u^{[i]}$ and $u^{[0]} = 1$ and

– multiplication : $u^{[i]} \wedge u^{[j]} = \frac{(i+j)!}{i!j!} u^{[i+j]}$
– Frobenius : $\phi(u^{[i]}) = p^i u^{[i]}$
– monodromy : $N(u^{[i]}) = u^{[i-1]}$

Definition 2.3. — The rigid Hyodo–Kato cohomology for Y/k semistable is given by $R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}\,(Y) := R\Gamma(\mathfrak{Z}, \omega^\bullet_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[u])$ with endomorphisms φ and N, such that $N\varphi = p\varphi N$.

This is justified by the following commutative diagram :

$$\begin{array}{ccccc} R\Gamma(\mathfrak{Z}, \omega^\bullet_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[u]) & \longrightarrow & R\Gamma(\mathfrak{Z}, \omega^\bullet_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}}[\![u]\!]) & \xrightarrow[u^{[i]}\mapsto 0]{\sim} & R\Gamma(\mathfrak{Z}, \omega^\bullet_{\mathcal{Z}/\mathcal{S},\mathbb{Q}}) \\ \downarrow \wr & & \downarrow & & \downarrow \\ R\Gamma(\mathfrak{Y}, \widetilde{\omega}^\bullet_{\mathcal{Y},\mathbb{Q}}[u]) & \xrightarrow{\sim} & R\Gamma(\mathfrak{Y}, \widetilde{\omega}^\bullet_{\mathcal{Y},\mathbb{Q}}[\![u]\!]) & \xrightarrow[u^{[i]}\mapsto 0]{\sim} & R\Gamma(\mathfrak{Y}, \omega^\bullet_{\mathcal{Y}/W^0,\mathbb{Q}}) \end{array}$$

Definition 2.4. — We set

$$\begin{array}{rcl} R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}\,(\mathcal{X}, \pi) & := & R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}\,(Y); \\ R\Gamma_{\mathrm{dR}}\,(\mathcal{X}) & := & R\Gamma(\mathfrak{X}, \omega^\bullet_{\mathcal{X}/V^{\sharp},\mathbb{Q}}). \end{array}$$

and define for a uniformiser $\pi \in V$ and $q \in \mathfrak{m}\backslash\{0\}$

$$\Psi_{\pi,q} : R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(\mathcal{X},\pi) \to R\Gamma_{\mathrm{dR}}(\mathcal{X})$$

induced by the natural morphism $\omega^{\bullet}_{\mathcal{Z}/W^{\varnothing},\mathbb{Q}} \to \omega^{\bullet}_{\mathcal{Z}/\mathcal{S},\mathbb{Q}} \to \omega^{\bullet}_{\mathcal{X}/V^{\sharp},\mathbb{Q}}$ and $\Psi_{\pi,q}(u^{[i]}) := \frac{(-\log_q(\pi))^i}{i!}$.

So the diagram now looks like :

$$\begin{array}{ccccc} & & R\Gamma_{\mathrm{rig}}(Y/W^{\varnothing}) & & \\ & \swarrow \quad \circlearrowleft & \downarrow & \circlearrowleft \quad \searrow & \\ R\Gamma_{\mathrm{rig}}(Y/W^{0}) & \xleftarrow{\sim} & R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y) & \xrightarrow{\Psi_{\pi,q}} & R\Gamma_{\mathrm{rig}}(Y/V^{\sharp},\pi), \\ & \nwarrow \quad \circlearrowleft & \downarrow & (*) \quad \nearrow & \\ & & R\Gamma_{\mathrm{rig}}(Y/\mathcal{S}) & & \end{array}$$

where all triangles except for $(*)$ commute. The triangle $(*)$ commutes if $q = \pi$.

Theorem 2.5. — *(Ertl–Yamada)*

(i) $\Psi_{\pi,q}$ *is independent o the choice of a uniformiser, that is for two uniformisers* $\pi, \pi' \in V$ *we have*

$$\Psi_{\pi,q} = \Psi_{\pi',q}.$$

(ii) *It depends on the choice of a branch of the p-adic logarithm* $\log_q$*, that is for* $q, q' \in \mathfrak{m}\backslash\{0\}$

$$\Psi_{\pi,q} = \Psi_{\pi,q'} \circ \exp(-\frac{\log_q(q')}{\mathrm{ord}_p(q')}N).$$

(iii) *For any* π *and* q,

$$\psi_{\pi,q} \otimes K : R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(\mathcal{X})_K \to R\Gamma_{\mathrm{dR}}(\mathcal{X}_K)$$

is a quasi-isomorphism.

(iv) *For a choice of uniformiser* $\pi \in V$, $\Psi_{\pi,\pi}$ *is compatible with the maps* Ψ^{HK}_{π} *of Hyodo–Kato and* Ψ^{GK}_{π} *of Große-Klönne.*

(v) *If* Y *has a compactification* $\overline{Y}$ *by a strictly semistable scheme with horizontal divisor, there is a rigid Hyodo–Kato theory of* Y *with compact support such that Poincaré duality is satisfied.*

Next we will see how this night help us in the conjecture of Flach-Morin.

3. A special case of Flach–Morin's conjecture

As we have seen, rigid Hyodo–Kato theory provides a good replacement of crystalline Hyodo–Kato theory in the non-proper (strictly) semistable case. Thus we may ask :

Question 3.1. — *If* Y *is (strictly) semistable but not necessarily proper, is it possible to obtain an exact triangle*

(2) $R\Gamma_{\mathrm{rig}}(Y^{\varnothing}/W(k)^{\varnothing}) \to \left[R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y) \xrightarrow{N} R\Gamma^{\mathrm{rig}}_{\mathrm{HK}}(Y)(-1)\right] \to R\Gamma^{*}_{\mathrm{rig},c}(Y^{\varnothing}/W(k)^{\varnothing})(-d-1)[-2d-1] \to$

in the derived category of φ*-modules ?*

For the rest of this subsection :

Y – a strictly semistable log scheme over k^0 (i.e. endowed with the canonical log structure).

Lemma 3.2. — *The natural morphism*

$$R\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^{\varnothing}) \to \left[R\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(Y) \xrightarrow{N} R\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(Y)(-1)\right]$$

between the "absolute" rigid cohomology of Y and the homotopy limit of the monodromy is a quasi-isomorphism of φ-modules.

Démonstration. — This can be shown by a local computation in which case it follows immediately from the definition of the monodromy on Kim–Hain complexes. □

Thus instead of (3) we suggest to study the existence of a triangle

$$R\Gamma_{\mathrm{rig}}(Y^{\varnothing}/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig},c}^{*}(Y^{\varnothing}/\mathscr{O}_F^{\varnothing})(-d-1)[-2d-1] \to \tag{3}$$

Indeed, there is a natural morphism $R\Gamma_{\mathrm{rig}}(Y^{\varnothing}/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^{\varnothing})$. However, we are looking for a natural interpretation of the last term in the triangle and the remaining maps.

In the compactifiable case, there is indeed such a natural interpretation of the second map in (3) given by our log rigid cohomology with compact support.

Thus we consider now the following situation :

- $\overline{Y}$ – a proper strictly semistable log scheme over k^0 with horizontal divisor ;
- D – the horizontal divisor ;
- Y – the complement $\overline{Y}\backslash D$, considered as a strictly semistable log scheme over k^0 ;
- $\overline{Y}^D$ – the log scheme which has the same underlying scheme as $\overline{Y}$ but log structure coming only from D.

Then we have

$$\begin{aligned} R\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^{\varnothing}) &\cong R\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^{\varnothing}) \\ R\Gamma_{\mathrm{rig}}(Y^{\varnothing}/\mathscr{O}_F^{\varnothing}) &\cong R\Gamma_{\mathrm{rig}}(\overline{Y}^D/\mathscr{O}_F^{\varnothing}) \\ R\Gamma_{\mathrm{rig},c}(Y^{\varnothing}/\mathscr{O}_F^{\varnothing}) &\cong R\Gamma_{\mathrm{rig},c}(\overline{Y}^D/\mathscr{O}_F^{\varnothing}), \end{aligned}$$

where the latter one is compactly supported towards D (analogous to Tsuji's crystalline definition). Now there is a canonical morphism

$$R\Gamma_{\mathrm{rig},c}(\overline{Y}^D/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig},c}(\overline{Y}/\mathscr{O}_F^{\varnothing}).$$

Together with Poincaré duality we obtain a morphism

$$R\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^{\varnothing}) \cong R\Gamma_{\mathrm{rig},c}(\overline{Y}/\mathscr{O}_F^{\varnothing})^*(-d-1)[-2d-1] \to R\Gamma_{\mathrm{rig},c}(\overline{Y}^D/\mathscr{O}_F^{\varnothing})^*(-d-1)[-2d-1].$$

So we obtain indeed natural maps and a **candidate** of an exact triangle

$$\begin{array}{ccccccc} R\Gamma_{\mathrm{rig}}(\overline{Y}^D/\mathscr{O}_F^{\varnothing}) & \longrightarrow & R\Gamma_{\mathrm{rig}}(\overline{Y}/\mathscr{O}_F^{\varnothing}) & \longrightarrow & R\Gamma_{\mathrm{rig},c}^{*}(\overline{Y}^D/\mathscr{O}_F^{\varnothing})(-d-1)[-2d-1] & \longrightarrow \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr & \\ R\Gamma_{\mathrm{rig}}(Y^{\varnothing}/\mathscr{O}_F^{\varnothing}) & \longrightarrow & R\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^{\varnothing}) & & R\Gamma_{\mathrm{rig},c}^{*}(Y^{\varnothing}/\mathscr{O}_F^{\varnothing})(-d-1)[-2d-1] & \longrightarrow \end{array}$$

Interestingly, in the lower conjectural triangle, no reference to a compactification appears, it is only in the construction of the second map that it appears. While it is not obvious how to show exactness of this triangle in the general case, I will now sketch in a specific situation how to obtain such a triangle in a more classical way.

For this we assume

- X – a k-variety of dimension $d+1$, such that Y is a simple normal crossing divisor ; endow X with the log structure induced by Y.

Berthelot showed [**3**, (2.3.1)], that we have a rigid localisation triangle

$$R\Gamma_{Y,\mathrm{rig}}(X^{\varnothing}/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig}}(X^{\varnothing}/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig}}((X\backslash Y)^{\varnothing}/\mathscr{O}_F^{\varnothing}) \to$$

Lemma 3.3. — *The natural morphism*

$$R\Gamma_{\mathrm{rig}}(X/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig}}((X\backslash Y)^{\varnothing}/\mathscr{O}_F^{\varnothing})$$

is a quasi-isomorphism

Démonstration. — It suffices to prove this locally. Then this is a special case of a theorem due to Tsuzuki [**12**, Thm. 3.5.1]. □

Thus we obtain an exact triangle

$$R\Gamma_{Y,\mathrm{rig}}(X^{\varnothing}/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig}}(X^{\varnothing}/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig}}(X/\mathscr{O}_F^{\varnothing}) \to$$

Lemma 3.4. — *The natural morphisms*

$$R\Gamma_{\mathrm{rig}}(X/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^{\varnothing}) \qquad \text{and} \qquad R\Gamma_{\mathrm{rig}}(X^{\varnothing}/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig}}(Y^{\varnothing}/\mathscr{O}_F^{\varnothing})$$

induce a quasi-isomorphism of homotopy limits

$$\left[R\Gamma_{\mathrm{rig}}(X^{\varnothing}/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig}}(X/\mathscr{O}_F^{\varnothing})\right] \cong \left[R\Gamma_{\mathrm{rig}}(Y^{\varnothing}/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^{\varnothing})\right].$$

Démonstration. — This follows from a local computation. □

This means that in the exact triangle above, we can replace $R\Gamma_{\mathrm{rig}}(X^{\varnothing}/\mathscr{O}_F^{\varnothing})$ by $R\Gamma_{\mathrm{rig}}(Y^{\varnothing}/\mathscr{O}_F^{\varnothing})$ and $R\Gamma_{\mathrm{rig}}(X/\mathscr{O}_F^{\varnothing})$ by $R\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^{\varnothing})$ and obtain an exact triangle

$$R\Gamma_{Y,\mathrm{rig}}(X^{\varnothing}/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig}}(Y^{\varnothing}/\mathscr{O}_F^{\varnothing}) \to R\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^{\varnothing}) \to$$

By Poincaré duality for usual rigid cohomology [**4**] we have a quasi-isomorpism

$$R\Gamma_{Y,\mathrm{rig}}(X^{\varnothing}/\mathscr{O}_F^{\varnothing}) \cong R\Gamma_{\mathrm{rig}\,,c}(Y^{\varnothing}/\mathscr{O}_F^{\varnothing})^*(-d-1)[-2d-2].$$

and together with the quasi-isomorphism

$$R\Gamma_{\mathrm{rig}}(Y/\mathscr{O}_F^{\varnothing}) \cong \left[R\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(Y) \xrightarrow{N} R\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(Y)(-1)\right]$$

we obtain :

Proposition 3.5. — *Let X be a smooth k-variety of dimension $d+1$ and $Y \subset X$ a simple normal crossing divisor. There is an exact triangle*

$$R\Gamma_{\mathrm{rig}}(Y^{\varnothing}/\mathscr{O}_F^{\varnothing}) \to \left[R\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(Y) \xrightarrow{N} R\Gamma_{\mathrm{HK}}^{\mathrm{rig}}(Y)(-1)\right] \to R\Gamma_{\mathrm{rig}\,,c}^{*}(Y^{\varnothing}/\mathscr{O}_F^{\varnothing})(-d-1)[-2d-1] \to$$

4. Approach via prismatic cohomology

As we have seen, the problem is far from solved. Here I would like to suggest a possible line of approach that is maybe different from what I have described before – or maybe an extension – namely via prismatic cohomology introduced by Bhatt and Scholze [**5**].

Prismatic cohomology can be seen as a theory that unifies various cohomology theories that are of interest in p-adic Hodge theory or more generally in p-adic geometry. The question is, whether one can take advantage of this for the problem that I presented today.

Question 4.1. — *How can we interpret the cohomology theories that occur in Flach–Morin's conjecture in the context of prismatic cohomology ? Does this allow to solve or even generalise the conjecture ?*

4.1. Overconvergent prismatic cohomology. — In the case of a proper smooth variety X over k, we have a comparison between the rigid cohomology $R\Gamma_{\mathrm{rig}}(X/F)$ and the prismatic cohomology $R\Gamma_{\Delta}(X/W(k))_{\mathbb{Q}}$ via crystalline cohomology $R\Gamma_{\mathrm{cris}}(X/W(k))_{\mathbb{Q}}$. However, as far as I know there is no direct comparison, or even a prismatic construction of rigid cohomology.

But one can imagine an overconvergent prismatic site, that in the case of a crystalline base prism computes rigid cohomology directly, even for open (smooth) varieties.

Steps :

(i) Construction of the overconvergent prismatic site in appropriate cases.
  (a) For crystalline base prisms.
  (b) For more general base prisms.

(ii) Comparison theorems in appropriate cases.
  (a) In the Monsky–Washnitzer situation : to (integral) Monsky–Washnitzer cohomology.
  (b) Globalise to (smooth) possible open varieties : to rigid cohomology.

(iii) Study prismatic isocrystals ?

4.2. Prismatic Hyodo–Kato theory. — As we have seen before a Hyodo–Kato theory (for a scheme over V, or a K-variety is a cohomology theory of finite dimensional F-vector spaces, with Frobenius and monodromy operator, and a Hyodo–Kato morphism to de Rham cohomology.

Using logarithmic prismatic cohomology developed by Koshikawa [**9**], we can indeed construct a cohomology theory of F-vector spaces. We also can obtain a Frobenius on these F-vector spaces. However, it is not a priori clear how to obtain, even in the situation of a crystalline prism, a monodromy operator and to which situations this can be generalised. Concerning the Hyodo–Kato map, considering the unifying properties of prismatic cohomology, one could hope that it allows a direct construction of the Hyodo–Kato map, maybe independent of the choice of a uniformiser.

(i) Monodromy :
  (a) Construction in the case of a crystalline base prism.
  (b) Construction on more general base prisms ?

(ii) Prismatic construction of the Hyodo–Kato map.

Références

[1] Berthelot B. — *Cohomologie cristalline des schémas de caractéristique $p > 0$*. Lecture Notes in Math., vol. 407, Springer-Verlag, 1974.

[2] Berthelot B. — *Cohomologie rigide et cohomologie rigide à supports propres : Première partie*. Prépublication de l'IRMAR, pp. 96-03, (1996).

[3] Berthelot B. — *Finitude et pureté cohomologique en cohomologie rigide (with an appendix in English by A.J. de Jong)*. Invent. Math., vol. 128, pp. 329–377, (1997).

[4] P. Berthelot — *Dualitd́e Poincar' et formule de Künneth en cohomologie rigide*. C.R. Acad. Sci. Paris vol. 325, pp. 493–498, (1997).

[5] Bhatt B., Scholze P. — *Prisms and prismatic cohomology*. Preprint, arXiv :1905.08229.

[6] B. Chiarellotto, N. Tsuzuki — *Clemens–Schmid exact sequence in characteristic p*. Math. Ann. vol. 358(3-4), pp. 971–1004, (2014).

[7] M. Flach, B. Morin — *Weil-étale cohomology and Zeta-values of proper regular arithmetic schemes*. Documenta Mathematica, vol. 23, pp. 1425–1560, (2018).

[8] Hyodo O., Kato K. — *Semi-stable reduction and crystalline cohomology with logarithmic poles*. Astérisque, vol. 223, pp. 221–268, (1994).

[9] Koshikawa T. — *Logarithmic prismatic cohomology I*. Preprint, arXiv :2007.14037.

[10] Morrison D.R — *The Clemens–Schmid exact sequence and applications.* In *Topics in transcendental algebraic geometry* (Princeton, N.J., 1981/1982), volume 106 of Ann. of Math. Stud., pages 101–119. Princeton Univ. Press, Princeton, NJ, 1984.

[11] Peters C.A.M., Steenbrink J.H.M. — *Mixed Hodge structures* Volume 52 of *Ergebnisse der Mathematik und ihrer Grenzgebiete, 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics].* Springer-Verlag, Berlin, 2008.

[12] N. Tsuzuki — *On the Gysin isomorphism of rigid cohomology.* Hiroshima Math. J. vol. 29, pp. 479–527, (1999).

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