The topic of our second meeting this semester is **Catalan's conjecture** - which is now known as **Mihăilescu's theorem**. It concerns a so called "Diophantine equation" - these are polynomial equations, where only integer solutions are studied. Such equations have been studied throughout the history. They often present problems which are easy to understand but hard to solve. This is also the case for Catalan's conjecture.

It was conjectured by **Eugène Charles Catalan** in 1844 and proven by **Preda Mihăilescu** in 2002. It states that there are no consecutive powers of natural numbers except 8 and 9:

Conjecture. *The only solution in natural numbers of the equation*

$$x^a - y^b = 1$$

for $a, b > 1$ *and* $x, y > 0$ *ist*

$$x = 3, \quad a = 2, \quad y = 2, \quad b = 3.$$

While the statement is easy to understand, Mihăilescu's proof contains tools of modern mathematics. In the short time that we have, we will certainly not be able to seriously study his proof. But we can look at earlier attempts and advancements and maybe some of Mihăilescu's methods, or speculate about further possible conjectures and questions.

As always, I encourage you to take up your own research beforehand, so that we can have a lively conversation. Any contribution is welcome, be it a specific question, a worked out example, a technical detail that you find interesting, a mathematical fact that is surprising, considerations about society and security,...
It doesn't have to be perfect, simply let your curiosity and creativity lead you.

Don't worry, if you don't know much about number theory or if you don't know where to start. Here are some suggestions:

(a) I nice short video about Catalan's conjecture: `https://www.youtube.com/watch?v=Us-__MukH9I`

(b) What are some "near misses" of Catalan's conjecture? `https://www.theoremoftheday.org/NumberTheory/Catalan/TotDCatalan.pdf`

(c) Here is survey article about the proof: `https://www.ams.org/journals/bull/2004-41-01/S0273-0979-03-00993-` `S0273-0979-03-00993-5.pdf` What are some of the key techniques?

(d) Here is a rather detailed outline of some of the special cases and the proof itself: `https://www.math.leidenuniv.nl/scripties/Daems.pdf`

- The case $b = 2$: Can you explain Lebesgue's proof that $x^a - y^2 = 1$ has no non-trivial integer solution for any prime a?
- The case $a = 2$ and $b \geqslant 5$: Ko Chao proofed in 1965 that $x^2 - y^b = 1$ has not non-trivial solution for any prime $b \geqslant 5$. Can you explain the proof.
- The case $a = 2$ and $b = 3$: Euler proofed that the only non-trivial solution in positive integers of $x^2 - y^3 = 1$ is $x = 3, y = 2$.
  What does this have to do with elliptic curves?
  What is the method of infinite descent?
- Cyclotomic fields play an important role in Mihăilescu's proof. What are they?
- What does Cassel's theorem say?

(e) There are several generalisations of Catalan's conjecture:

- It is a conjecture that for every natural number n, there are only finitely many pairs of perfect powers with difference n. Choose a natural number n (maybe small "enough") and find two perfect powers that differ by n. Try to write a small program for this.
- What is Pillai's conjecture? `https://math.stackexchange.com/questions/1642096/special-case-of-pil`
- What is the Fermat-Catalan conjecture?

(f) Here is a nice and amusing discussion by Ribenboim about Catalan's conjecture: `http://www.numdam.org/article/SPHM_1994___6_A1_0.pdf` He discusses some cases, and muses about methods to solve it. (This is from the nineties, before Catalan's conjecture was proved.

(g) The equation $x^2 - y^3 = 1$ is a socalled Mordell's equation. https://math.stackexchange.com/questions/1898338/find-values-of-x-and-y-when-x2-y3-1 What is a Mordell's equation?

(h) On the other hand, we can look at the equations $2^a - 3^b = 1$ or $3^a - 2^b = 1$. Gersonides in the 13th century. How can we treat these cases?
https://math.stackexchange.com/questions/958304/distance-between-powers-of-2-and-3
https://math.stackexchange.com/questions/1642096/special-case-of-pillais-conjecture