# The de Rham and de Rham-Witt complex

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# 1 Prelimiaries

## 1.1 The étale and Nisnevich site

We will briefly recall the small étale and Nisneviche site of a scheme $X$.

**Definition 1.1.1.** A morphism of schemes $f : Y \to X$ is étale if it is flat and unramified (in particular it is of finite type). It is called completey decomposed, if in addition for every point $x \in X$ there is a point $y \in f^{-1}(x)$ such that the induced morphism on residue fields $k(x) \to k(y)$ is an isomorphism.
A family of morphisms $\{f_i : X_i \to X\}_{i \in I}$ form a Nisnevich cover if the $f_i$ are étale, and for each $x \in X$ there is $i \in I$ such that $f_i$ is completely decomposed at $x$.

Thus any Zariski covering is Nisnevich and any Nisnevich cover is étale. The property of being completely decomposed is stable under pull-backs: if in a cartesian square

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{\alpha} & X \end{array}$$

the morphism $f : U \to X$ is completely decomposed at $x \in X$ then $g : V \to Y$ is completely decomposed at $y = \alpha^{-1}(x)$. From this we deduce the following. Let for a scheme $X$ and $\mathscr{C}$ be a full subcategory of $\mathrm{Sch}/X$ such that the pull-back in $\mathrm{Sch}/X$ of the diagram in $\mathscr{C}$

$$\begin{array}{ccc} & & U \\ & & \downarrow{\scriptstyle p} \\ Y & \longrightarrow & X \end{array}$$

where $p$ is étale, is again in $\mathscr{C}$, then the Nisnevich coverings form a basis for a (Grothendieck) topology on $\mathscr{C}$. The Nisnevich topology is finer than the Zariski topology but coarser than the étale topology. As in the étale case, we can define the small and the big Nisnevich site of a noetherian scheme $X$. We denote the small étale resp. Nisnevich site of $x$ by $X_{\text{ét/Nisn}}$. The local rings of a scheme with its Nisnevich topology are Henselian rings, while the local rings with respect to the étale topology are strictly Henselian rings. There is a morphism of sites

$$\varepsilon : X_{\text{ét}} \to X_{\text{Nis}}.$$

Some properties:

1. **Subcanonical**. The Nisnevich topology is (as the étale and Zariski topologies) sub-canonical: every representable pre-sheaf is in fact a sheaf.

2. **Stratification**. Let $\{f_i : X_i \to X\}_{i\in I}$ be a Nisnevich cover. Then there is a strictly decreasing sequence of closed subsets
$$X \supsetneq Z_{j_0} \supsetneq \cdots \supsetneq Z_{j_{n+1}} = \varnothing$$
such that $\{j_0, \ldots, j_n\} \subset I$ and for each $m$ $f_{j_m}\big|_{Z_{j_m} - Z_{j_{m+1}}}$ admits a section.

**Definition 1.1.2.** A cartesian diagram
$$\begin{array}{ccc} U \times V & \longrightarrow & V \\ \downarrow & & \downarrow p \\ U & \xrightarrow{\ i\ } & X \end{array}$$
such that $i$ is an open immersion, $p$ is étale and for $Z = X - U$ p induces an isomorphism $p^{-1}(Z) \to Z$ is called distinguished or elementary. Hence we call the covering family $\{i, p\}$ of $X$ elementary.

The stratification lemma implies that the elementary coverings form a basis of the Nisnevich topology. To see if a presheaf is a sheaf it satisfies therefore to verify for the elementary coverings.

3. **Mayer-Vietoris**. Furthermore, there is a Mayer-Vietoris sequence for an elementary covering.

4. **Descent spectral sequence**. One of the key properties of the Nisnevich site is the existence of a descent (local-to-global) spectral sequence for the Quillen $K$-theory of coherent sheaves.

5. **Cohomological dimension**. Let $X$ be noetherian, quasi-separated of dimension $d$. Then for any abelian sheaf $\mathcal{F}$ on $X$
$$\mathrm{H}^n_{\mathrm{Nis}}(X, \mathcal{F}) = 0 \qquad \text{if} \qquad n > d.$$

## 1.2 Some notions from homological algebra

The following is not very precise. For a small site $\mathbb{S}$ let

1. $\mathrm{Sh}(\mathbb{S})$ be the category of sheaves of abelian groups on $\mathbb{S}$.
2. $C(\mathbb{S})$ be the category of unbounded complexes.
3. $\mathrm{Sh}_{\mathrm{pro}}(\mathbb{S})$ be the category of pro-systems in $\mathrm{Sh}(\mathbb{S})$.
4. $C_{\mathrm{pro}}(\mathbb{S})$ be the category of pro-systems in $C(\mathbb{S})$.
5. $K_{\mathrm{pro}}(\mathbb{S})$ the homotopy category of $C_{\mathrm{pro}}(\mathbb{S})$.
6. $D_{\mathrm{pro}}(\mathbb{S})$ the Verdier quotient of $K_{\mathrm{pro}}(\mathbb{S})$.

The idea in general is: one starts with a category of complexes. It is sufficient to consider them up to homotopy (as we want to go to cohomology in the end). And then invert by brute force the quasi-isomorphisms. As a consequence one obtains an additive category. However, in general it is not abelian. Therefore on has to replace the concept of short exact sequences by exact triangles. A category with a translation functor and a clasS of triangles (called distinguished) which satisfiey four basic properties is called a triangulated category. The homotopy and derived category are both triangulated.

We denote furthermore by $\mathrm{S}(\mathbb{S})$ the closed simplicial model category of simplicial presheaves on $\mathbb{S}$, where cofibrations are injective morphisms of pre-sheaves and weak equivalences are those maps which induce isomorphisms on homotopy sheaves. We endow the category of unbounded complexes $C(\mathbb{S})$ as well with a closed simplicial model structure. Similar to the definitions above we have categories $\mathrm{S}_{\mathrm{pro}}(\mathbb{S})$ and $C_{\mathrm{pro}}(\mathbb{S})$. The structures are called pro-model structures. They are due to Isaksen. We allow only $\mathbb{N}$ as index category, so only countable inverse limits and finite direct limits exist.

For a scheme $X$ we let $\mathrm{S}_{\mathrm{pro}}(X)_{\text{ét/Nis}} = \mathrm{S}_{\mathrm{pro}}(X_{\text{ét/Nis}})$ etc.

### 1.3 Adic rings/spaces and formal schemes

An adic noetherian ring $R$ is a noetherian ring which has a topological basis generated by neighbourhoods of zero $\{I^n\}_{n\in\mathbb{N}}$ where $I \subset R$ is an ideal, such that $R$ as a topological space is Hausdorff and complete. Such an ideal is called an ideal of definition. An ideal $J \subset R$ is an ideal of definitino iff it is open and its poers tend to zero. For the choice of an ideal of definition $I$ we call the associated topology $I$-adic, and the descending filtration defined by powers of $I$ is the $I$-adic filtration.

For an adic noetherian ring $R$, we can define its formal spectrum $\operatorname{Spf} R$. For an ideal of definition $I$, e define

$$\operatorname{Spf} R = \operatorname{colim} \operatorname{Spec}(R/I^n).$$

As $I$ is nilpotent, the underlying topological space is $\operatorname{Spec} R/I$, and it contains all closed points of $\operatorname{Spec} R$. The structure sheaf is given by

$$\mathscr{O}_R = \varprojlim \mathscr{O}_{R/I^n},$$

where the limit is taken in the category of topological rings. The formal spectrum depends only on the underlying ring $R$ and not on the choice of an ideal of definition. This notion can be globalised to formal schemes.

### 1.4 The de Rham-Witt complex

The de Rham-Witt complex is a shef on a scheme over a perfect field of characteristic $p$ (or more generally of a $\mathbb{Z}_{(p)}$-algebra. It provides a complex which is explicit and (relatively) computable. Its hypercohomology gives the crystalline cohomology. It is a pro-system of differentially graded algebras.

The de Rham-Witt complex over a scheme $X$ of characteristic $p$ can be defined as the initial object of the category of Witt complexes over $X$. It is a universal object in the category of projective systems of differentially graded algebras the extends the sheaf of Witt vectors and satisfies certain relations ith respect to Frobenius and Verschiebung. It is uniquely defined by the following properties:

1. In degree zero it is isomorphic to the ring of Witt vectors $W.\Omega^0_X = W.\mathscr{O}_X$.
2. For $x \in W_n\Omega^i$ and $y \in W_n\Omega^j$ it satisfies the relation $V(xdy) = (Vx)dVy$.
3. For $n \geqslant 1$, $x \in W_1\Omega^0$ and $y \in W_n\Omega^0$ one has $(Vy)dx = V(x^{p-1})dVx$.

It is constructed inductively as quotients of de Rham complexes over $W_n\mathscr{O}_X$ devided by the obvious relations (only involving Verschiebung $V$ and restriction $R$). Then one checks that it satisfies also the desired properties ith respect to the induced Frobenius map.

## 2 Cristalline and de Rham cohomology

Let $k$ be a perfect field of characteristic $p > 0$ and $S = W(k)$ the ring of Witt vectors which is an adic ring with ideal of definition $I = (p)$. Let $X. \in \mathrm{Sch}_{S.}$ (a $p$-adic formal scheme over the Witt vectors). We denote $X_n = X. \otimes W_n(k)$. Then in particular $X_1$ is its special fiber. Note that the étale/Nisnevich sites of all $X_i$ are isomorphic.

**Definition 2.0.1.** 1. Let

$$\Omega^*_{X.} \in C_{\mathrm{pro}}(X_1)_{\text{ét/Nis}}$$

be the pro-system of de Rham complexes $\Omega^*_{X_n/W_n(k)}$.

2. Let

$$W.\Omega^*_{X_1} \in C_{\mathrm{pro}}(X_1)_{\text{ét/Nis}}$$

the pro-system of de Rham-Witt complexes.

For $n \in \mathbb{N}$ denote

$$d\log : \mathscr{O}_X^* \to W_n\Omega_X^1$$

the morphism of abelian sheaves defined locally by $x \mapsto \frac{d[x]}{[x]}$. This induces a morphism of projective systems

$$d\log : \mathscr{O}_X^* \to W_\bullet\Omega_X^1.$$

Let $W_n\Omega^i_{X,\log} \subset W_n\Omega^i_X$ be the sub-sheaf generated étale(or Nisnevich)-locally by sections of the form $d\log[x_1]\dots d\log[x_i]$ for $x_j \in \mathscr{O}_X^*$. This construction is known to be functorial in $X$, and the product structure of $W_n\Omega^\bullet_X$ carries over to $W_n\Omega^\bullet_{X,\log}$. For $n$ variable, $W_\bullet\Omega^\bullet_{X,\log}$ is an abelian sub-pro-sheaf of $W_\bullet\Omega^\bullet_X$ and we set $W\Omega^\bullet_{X,\log} := \varprojlim W_\bullet\Omega^\bullet_{X,\log}$. For $i \in \mathbb{N}_0$ there is a short exact sequence of pro-systems for étale topology

$$0 \to W_\bullet\Omega^i_{X,\log} \to W_\bullet\Omega^i_X \xrightarrow{\mathrm{F}-1} W_\bullet\Omega^i_X \to 0$$

where F denotes a lift of the Frobenius endomorphism.

For the morphism of sites

$$\varepsilon : X_{\text{ét}} \to X_{\text{Nis}}$$

we can identify

$$\varepsilon^* W_n\Omega^r_{X,\text{Nis}} = W_n\Omega^r_{X,\text{ét}}$$

and Kato shows

**Proposition 2.0.2.** *The natural map*

$$W_n\Omega^r_{X,\log,Nis} \to \varepsilon_* W_n\Omega^r_{X,\log;\acute{e}t}$$

*is an isomorphism.*

This means that $\varepsilon_* W_n\Omega^r_{X,\log;\text{ét}}$ is Nosnevich locally generated by symbols.

We introduce some subcomplexes of the de Rham and de Rham-Witt complexes which will play an important role in the obstruction theory.

**Definition 2.0.3.** For $r < p$ we define

$$p(r)\Omega^*_{X.} \in C_{\text{pro}}(X_1)_{\text{ét/Nis}}$$

as the complex

$$p^r\,\mathscr{O}_{X.} \to p^{r-1}\Omega^1_{X.} \to \cdots \to p\Omega^{r-1}_{X.} \to \Omega^r_{X.} \to \cdots$$

and

$$q(r)W.\Omega^*_{X_1} \in C_{\text{pro}}(X_1)_{\text{ét/Nis}}$$

as the complex

$$p^{r-1}VW.\,\mathscr{O}_{X_1} \to p^{r-2}VW.\Omega^1_{X_1} \to \cdots \to pVW.\Omega^{r-2}_{X_1} \to VW.\Omega^{r-1}_{X_1} \to W.\Omega^r_{X_1} \to \cdots$$

It is possible to define this for $r \geqslant p$ if one introduces divided powers, but this creates also some problems for example with syntomic cohomology.

We want to construct quasi-isomorphisms (or isomorphisms in $D_{\text{pro}}(X_1)$

$$\begin{aligned} \Omega^*_{X.} &\cong W.\Omega^*_{X_1} \\ p(r)\Omega^*_{X.} &\cong q(r)W.\Omega^*_{X_1} \end{aligned}$$

In order to do so, we make use as auxiliary tool a complex on the PD-envelop of $X.$.

Consider a closed embedding

$$X. \to Z.$$

into a smooth scheme over $W.(k)$ which allos a lift of Frobenius $F : Z. \to Z.$. For each $n$, we have the PD-envelope

$$X_n \to D_n = D_{X_n}(Z_n).$$

*Remark* 2.0.4. For a PD-algebra $(A, I, \gamma)$ and an $A$-albegra $B$ with some ideal $J$, there is a universal PD-algebra $D_A(B)$ such that its PD-structure is compatible ith $\gamma$ and $J \cdot D_A(B)$ is contained in its PD-ideal. This can be globalised. For a closed immersion of schemes we take $J$ to be the defining ideal. This defines the PD-envelope of the closed immersion.

$D_n$ is endowed with a de Rham complex

$$\Omega^*_{D_n/W_n} = \mathscr{O}_{D_n} \otimes \Omega^*_{Z_n/W_n}$$

such that for the PD-structure

$$d\gamma^n(x) = \gamma^{n-1}(x)dx.$$

*Remark* 2.0.5. A PD-sttructure is defined in a way such that $n! \cdot \gamma^n(x) = x^n$ thereby introducing devided powers.

Let $J_n$ be the defining ideal of $X_n \subset D_n$. Then $I_n = (J_n, p)$ is the ideal of $X_1 \subset D_n$. These ideals are nilpotent. We denote their devided powers by $J_n^{[j]}$ and $I_n^{[j]}$ respectively. If $j < p$ they coincide ith the usual powers.

By definition, the étale/Nisnevich sites of $X_1$ and $D_n$ coincide. For simplicity we assume $r < p$.

**Definition 2.0.6.** Define $J(r)\Omega^*_{D.} \in C_{\mathrm{pro}}(X_1)_{\text{ét}/\mathrm{Nis}}$ as the complex

$$J_.^r \to J_.^{r-1} \otimes \Omega^1_{Z.} \to \cdots \to J. \otimes \Omega^{r-1}_{Z.} \to \mathscr{O}_{D.} \otimes \Omega^r_{Z.} \to$$

and similarly $I(r)\Omega^*_{D.} \in C_{\mathrm{pro}}(X_1)_{\text{ét}/\mathrm{Nis}}$.

From now on, we assume that $X.$ is smooth over $W.$. Illusie shows in his proof of the comparison theorm (de Rham-Witt – crystalline) that for each $n$ the lifting of Frobenius

$$\Phi(F) : \mathscr{O}_{D_n} \to W_n \mathscr{O}_{X_1}$$

induces a quasi-isomorphism of differentially graded algebras

$$\Phi(F) : \Omega^*_{D_n} \to W_n\Omega^*_{X_1}.$$

Berthelot and Ogus show furthermore, that the restrictions

$$\begin{array}{rcl} \Omega^*_{D_n} & \to & \Omega^*_{X_n} \\ J(r)\Omega^*_{D_n} & \to & \Omega^{\geqslant r}_{X_n} \\ I(r)\Omega^*_{D_n} & \to & p(r)\Omega^*_{X_n} \end{array}$$

are also quasi-isomorphisms (or rather they show that the crystalline cohomology of $X$ and the de Rham cohomology over the PD-envelope of $X$ coincide). Thus we obtain a diagram

$$\begin{array}{ccccc} & & \Omega^*_{D.} & & \\ & \swarrow\sim & & \sim\searrow^{\Phi(F)} & \\ \Omega^*_{X.} & & & & W.\Omega^*_{X_1} \end{array}$$

which represents a morphism in $D_{\mathrm{pro}}(X_1)_{\text{ét}/\mathrm{Nis}}$. In particular, this shows that the complexes in question are canonically quasi-isomorphic. This is independent of the choice of $Z.$.

**Proposition 2.0.7.** *For $X.$ as before, the above diagram induces a diagram*

$$\begin{array}{ccccc} & & I(r)\Omega^*_{D.} & & \\ & \swarrow\sim & & \sim\searrow^{\Phi(F)} & \\ p(r)\Omega^*_{X.} & & & & q(r)W.\Omega^*_{X_1} \end{array}$$

*and therefore a canonical quasi-isomorphism*

$$p(r)\Omega^*_{X.} \to q(r)W.\Omega^*_{X_1}$$

*independent of the choice of $Z.$.*

PROOF: We mentioned above that $I(r)\Omega^*_{D_n} \to p(r)\Omega^*_{X_n}$ is a quasi-isomorphism. It remains to sho that $\Phi(F)$ is also a quasi-isomorphism.

Because of $I(r)\Omega^*_{D_n} \to p(r)\Omega^*_{X_n}$ we may also assume that $X. = Z. = D.$. Furthermore, this is a local problem, therefore we can assume that the involved schemes are affine with Frobenius lift $F$. Let $d$ be the dimension of $X_1$. For a sequence $\nu_* = \nu_0 \geqslant \cdots \geqslant \nu_{d+1} \geqslant 0$ such that $\nu_{i+1} \geqslant \nu_i - 1$ and $\nu_i < p$, and $\nu_{d+1} = \max(0, \nu_d - 1)$ we consider a subcomplex $q(\nu_*)W.\Omega^*_{X_1}$

$$q(\nu_*)W.\Omega^i_{X_1} = \begin{cases} p^{\nu_i} & \text{for } \nu_i = \nu_{i+1} \\ p^{\nu_{i+1}}VW.\Omega^i_{X_1} & \text{for } \nu_i = \nu_{i+1}+1 \end{cases}$$

Now $\Phi(F)$ induces a map

$$\Phi(F) : p^{\nu_*}\Omega^*_{X.} \to q(\nu_*)W.\Omega^*_{X_1}.$$

**Lemma 2.0.8.** *This map induces an isomorphism in $D_{pro}(X_1)_{\acute{e}t/Nis}$.*

PROOF: This is done by induction on $N = \sum \nu_i$. For $N = 0$ this means that

$$\Omega^*_{A.} \to W.\Omega^*_{A_1}$$

is a quasi-isomorphism, which is the comparison isomorphism by Illusie.
No assume the result for smaller values than $N > 0$. Let $i$ the smallest number such that $\nu_0 = \cdots = \nu_i > \nu_{i+1}$. Let $\mu_*$ such that $\mu_j = \nu_j$ for $j \geqslant i$ and $\mu_j = \nu_j - 1$ for $j < i$. By induction $p^{\mu_*}\Omega^*_{X.} \to q(\mu_*)W.\Omega^*_{X_1}$ is a quasi-isomorphism. The quotients are isomorphic to the following complexes

$$\begin{aligned} p^{\mu_*}\Omega^*_{X.}/p^{\nu_*}\Omega^*_{X.} &\cong \mathscr{O}_{X_1} \to \cdots \to \Omega^i_{X_1} \\ q(\mu_*)W.\Omega^*_{X_1}/q(\nu_*)W.\Omega^*_{X_1} &\cong W(X_1)/pW(X_1) \to \cdots \to W.\Omega^i_{X_1}/VW.\Omega^i_{X_1} \end{aligned}$$

Illusie shoed that the right-hand sides are quasi-isomorphic and this shos the lemme. □

As $q(\nu_*)$ is a general case of the subcomplex $q(r)W.\Omega^*_{X_1}$ which corresponds to the sequence $\nu_i = \max(0, r - i)$, this finishes the proof of the proposition. □

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