Crystalline Hodge obstruction and motivic complex

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Recall that we want to prove that for k a perfect field of positive characteristic p, X/W a smooth projective scheme with closed fiber X_1 with $\dim X_1 = d$ and $p > d+6$, and for $\xi_1 \in K_0(X_1)_{\mathbb{Q}}$ the following are equivalent:

1. ξ_1 is of Hodge type (i.e. $\Phi^{-1} \circ \mathrm{ch}(\xi_1) \in \oplus_r F^r \mathrm{H}^{2r}_{\mathrm{dR}}(X/S)$).
2. There is $\widehat{\xi} \in \varprojlim K_0(X_n)_{\mathbb{Q}}$ such that $\widehat{\xi}\big|_{X_1} = \xi_1 \in K_0(X_1)_{\mathbb{Q}}$.

To this effect, we considered the following diagram:

$$\begin{tikzcd}
\oplus_{r\leqslant d} \mathrm{CH}^r_{\mathrm{cont}}(X_\bullet)_{\mathbb{Q}} \arrow[r] & \oplus_{r\leqslant d} \mathrm{CH}^r(X_1)_{\mathbb{Q}} \arrow[r, "\mathrm{Ob}"] & \oplus_{r\leqslant d} \mathrm{H}^{2r}_{\mathrm{cont}}(X_1, p(r)\Omega^{<r}) \\
K_0^{\mathrm{cont}}(X_\bullet)_{\mathbb{Q}} \arrow[u, "\Gamma"] \arrow[d] & & \\
\varprojlim K_0(X_n)_{\mathbb{Q}} \arrow[r] & K_0(X_1)_{\mathbb{Q}} \arrow[uu, hook, two heads] &
\end{tikzcd} \tag{1}$$

where the top sequence is exact.
This shows, that an element $\xi_1 \in K_0(X_1)_{\mathbb{Q}}$ can be lifted if and only the associated element in the Chow groups can be lifted, and this is controlled by the obstruction map. Now we have to link the behaviour of this obstruction map to the property of being of Hodge type. This is our goal today.
For this part, we assume that X_1/k is proper.

1 The obstruction sequence

We have already seen how to construct continuous cohomology groups. As "usual" Chow groups can be given in terms of motivic cohomology as

$$\mathrm{CH}^r(X) = \mathrm{H}^2\, r(X, \mathbb{Z}(r))$$

we copy this to define continuous Chow groups.

Definition 1.1. The continuous Chow group of $X_\bullet \in \mathscr{S}m_{W_\bullet}$ is given by

$$\mathrm{CH}^r_{\mathrm{cont}}(X_\bullet) := \mathrm{H}^{2r}_{\mathrm{cont}}(X_1, \mathbb{Z}_{X_\bullet}(r))$$

where $\mathbb{Z}_\bullet(r)$ is the motivic procomplex.

We have established the so-called motivic fundamental triangle in $D_{\mathrm{pro}}(X_1)$

$$\begin{tikzcd}
p(r)\Omega^{<r}_{X_\bullet}[-1] \arrow[r] \arrow[d, equal] & \mathbb{Z}_{X_\bullet}(r) \arrow[r] \arrow[d] & \mathbb{Z}_{X_1}(r) \arrow[r] \arrow[d, "d\log"] & \cdots \\
p(r)\Omega^{<r}_{X_\bullet}[-1] \arrow[r] & \mathfrak{S}_{X_\bullet}(r) \arrow[r] & W_\bullet\Omega^r_{X_1,\log}[-r] \arrow[r] & \cdots
\end{tikzcd}$$

Applying the continuous cohomology functor to the upper exact triangle gives rise to the exact obstruction sequence

$$\mathrm{CH}^r_{\mathrm{cont}}(X_\bullet) \to \mathrm{CH}^r(X_1) \xrightarrow{\mathrm{Ob}} \mathrm{H}^{2r}_{\mathrm{cont}}(X_1, p(r)\Omega^{<r}_{X_\bullet}). \tag{2}$$

The idea now is to compare the image of a cycle $\xi_1 \in \mathrm{CH}^r(X_1)$ under the obstruction map Ob to its crystalline cycle class.

Remark 1.2. In order to tie this in with a possible algebraisation, we note, that there is an exact sequence

$$0 \to \varprojlim^1 \mathrm{H}^{2r-1}(X_1, \mathbb{Z}_{X_n}(r)) \to \mathrm{CH}^r_{\mathrm{cont}}(X_\bullet) \to \varprojlim \mathrm{H}^{2r}(X_1, \mathbb{Z}_{X_n}(r)) \to 0$$

In particular, for $r = 1$ we have seen that by definition $\mathbb{Z}_{X_\bullet}(1) = \mathbb{G}_{m,X_\bullet}[-1]$ and $\varprojlim^1 \mathrm{H}^1(X_1, \mathbb{G}_{m,X_n})$ vanishes. Thus we obtain an isomorphism

$$\mathrm{CH}^1_{cont}(X_\bullet) \xrightarrow{\sim} \varprojlim \mathrm{Pic}(X_n)$$

If X is the p-adic formal scheme associated to a smooth projective scheme X/W there is an algebraisation isomorphism

$$\mathrm{Pic}(X) \xrightarrow{\sim} \varprojlim \mathrm{Pic}(X_n)$$

so if we can lift a cycle to the continuous Chow group, we can algebrais it. Unfortunately an analogue for higher degrees is unknown.

2 Crystalline cycle classes

Crystalline cycle classes were constructed by Gros using the logarithmic differential morphism

$$d\log \circ [-] : \mathscr{K}^M_r \to W_\bullet\Omega^r_{\log}.$$

For a closed integral subscheme $Y \subset X_1$ of codimension r, he shows a purity statement

$$\mathrm{H}^j_Y(X, W_n\omega^j_{X_1,\log}) = \mathrm{H}^{j+1}_Y(X, W_n\Omega^j_{X_1,\log}) = 0$$

for $j < r$. Via the long exact excision sequence for cohomology, this gives rise to a morphism

$$\mathbb{Z}\cdot[Y] = \mathrm{H}^r_Y(X_1, \mathscr{K}^M_r) \xrightarrow{d\log} \mathbb{Z}/p^\bullet \cdot [Y] = \mathrm{H}^r_Y(X_1, W_\bullet\Omega^r_{X_1,\log}).$$

The image of $[Y]$ in the cohomology without supports is the cycle class of Y. By linearity this can be extended to a cycle class map

$$\mathrm{CH}^r(X_1) \to \mathrm{H}^r_{cont}(X_1, W_\bullet\Omega^r_{\log}).$$

An easier way to obtain the cycle class map uses Gersten resolution of the Milnor K-sheaf. In [17] Kerz shows a Bloch formula for the Milnor K-sheaf

$$\mathrm{CH}^r(X_1) = \mathrm{H}^r(X_1, \mathscr{K}^M_r),$$

for schemes with infinite residue fields. This is done by comparing the Gerstne resolution of the Chow group, with the Gersten resolution established by Kerz for the Milnor K-sheaf. Then one makes use of the fact that the Bloch formula is known for fields. After [?, Kerz3]his can be generalised to schemes with finite residue fields. Then the log-differential morphism of pro-sheaves induces again a map

$$\mathrm{CH}^r(X_1) = \mathrm{H}^r(X_1, \mathscr{K}_r^M) \to \mathrm{H}^r_{cont}(X_1, W_\bullet\Omega^r_{\log}).$$

Now we recall that there is a natural map of procomplexes over X_1

$$W_\bullet\Omega^r_{\log}[-r] \to W_\bullet\Omega^{\geqslant r} \to q(r)W_\bullet\Omega^\bullet, \tag{3}$$

where $q(r)W_\bullet\Omega^\bullet = p^{r-1}VW_\bullet\mathscr{O} \to p^{r-2}VW_\bullet\Omega^1 \to \cdots \to pVW_\bullet\Omega^{r-2} \to VW_\bullet\Omega^{r-1} \to W_\bullet\Omega^r \to \cdots$ is the shifted de Rham-Witt complex.

Definition 2.1. For $\xi \in \mathrm{CH}^r(X_1)$ we define the refined crystalline cycle class

$$c(\xi) \in \mathrm{H}^{2r}_{cont}(X_1, q(r)W_\bullet\Omega^\bullet)$$

via the morphism of pro-sheaf complexes (3). The crystalline cycle class of ξ is the image of $c(\xi)$ in $\mathrm{H}^{2r}_{cont}(X_1, W_\bullet\Omega^\bullet)$. It is denoted by $c_{cris}(\xi)$.

3 Cycle classes of Hodge type

Recall that we established comparison isomorphisms in the derived Nisnevich category over X_1

$$\begin{aligned} q(r)W_\bullet\Omega^\bullet_{X_1} &\cong p(r)\Omega^\bullet_{X_\bullet} \\ W_\bullet\Omega^\bullet_{X_1} &\cong \Omega^\bullet_{X_\bullet}, \end{aligned}$$

Applying the continuous cohomology functor, we may identify

$$\begin{aligned} \mathrm{H}^i_{cont}(X_1, q(r)W_\bullet\Omega^\bullet) &= \mathrm{H}^i_{cont}(X_1, p(r)\Omega^\bullet_{X_\bullet}) \\ \mathrm{H}^i_{cont}(X_1, W_\bullet\Omega^\bullet) &= \mathrm{H}^i_{cont}(X_1, \Omega^\bullet_{X_\bullet}). \end{aligned}$$

Now we can say, what it means for a cycle to be of Hodge type in this context.

Definition 3.1.

1. The crystalline cycle class of ξ is of Hodge type if and only if $c_{cris}(\xi)$ lies in the image of $\mathrm{H}^i_{cont}(X_1, \Omega^{\geqslant r}_{X_\bullet})$ in $\mathrm{H}^{2r}_{cont}(X_1, \Omega^\bullet_{X_\bullet})$.
2. The refined crystalline cycle class of ξ is of Hodge type if and only if $c(\xi)$ lies in the image of $\mathrm{H}^i_{cont}(X_1, \Omega^{\geqslant r}_{X_\bullet})$ in $\mathrm{H}^{2r}_{cont}(X_1, p(r)\Omega^\bullet_{X_\bullet})$.
3. The crystalline cycle class of ξ is of Hodge type modula torsion if and only if $c_{cris}(\xi) \otimes \mathbb{Q}$ lies in the image of $\mathrm{H}^i_{cont}(X_1, \Omega^{\geqslant r}_{X_\bullet}) \otimes \mathbb{Q}$ in $\mathrm{H}^{2r}_{cont}(X_1, \Omega^\bullet_{X_\bullet}) \otimes \mathbb{Q}$.

The definition used in the original theorem which we want to prove is:

4. An element $\xi_1 \in K_0(X_1)_{\mathbb{Q}}$ is of Hodge type if and only if

$$\Phi^{-1} \circ \mathrm{ch}(\xi_1) \in \bigoplus_r F^r\,\mathrm{H}^{2r}_{dR}(X/W)$$

where Φ is the comparison morphism between de Rham and crystalline cohomology and ch is the crystalline Chern character defined by Gros.

As rationally we have an isomorphism

$$K_0(X_1)_{\mathbb{Q}} \cong \oplus\,\mathrm{CH}^r(X_1)_{\mathbb{Q}}$$

and moreover, the crystalline Chenr character on $K_0(X_1)$ and the crystalline cycle classes on the Chow groups $\mathrm{CH}^r(X_1)$ are compatible (due to the fact that both are constructed via the logarithmic differential map), we see, that the two definitions of Hodge type on the one hand for the K-group and on the other hand for the Chow groups are equivalent.

Remark 3.2. 1. By degeneration of the Hodge-de Rham spectral sequence up to torsion, the map $\mathrm{H}^{2r}_{cont}(X_1, \Omega^{\geqslant r}_{X_\bullet}) \otimes \mathbb{Q} \to \mathrm{H}^{2r}_{cont}(X_1, \Omega^\bullet_{X_\bullet}) \otimes \mathbb{Q}$ is injective.

2. If for all $a, b \in \mathbb{N}$ $\mathrm{H}^b_{cont}(X_1, \Omega^b_{X_\bullet})$ is torsion free as $W(k$-module, then the composition

$$\mathrm{H}^{2r}_{cont}(X_1, \Omega^{\geqslant r}_{X_\bullet}) \to \mathrm{H}^{2r}_{cont}(X_1, p(r)\Omega^\bullet_{X_\bullet}) \to \mathrm{H}^{2r}_{cont}(X_1, \Omega^\bullet_{X_\bullet})$$

is injective, and so is in particular th left morphism.

3. The map $\mathrm{H}^{2r}(X_1, p(r)\Omega^\bullet_{X_\bullet}) \otimes \mathbb{Q} \to \mathrm{H}^{2r}_{cont}(X_1, \Omega^\bullet_{X_\bullet}) \otimes \mathbb{Q}$ is an isomorphism. This statement will be important for the proof of the main theorem of this section.

4 The Hodge obstruction

We come now to prove one of the two main ingredients for the main theorem of the paper.

Theoreme 4.1. *Let $X_\bullet/W_\bullet$ be a smooth projective p-adic formal scheme. Let $\xi_1 \in \mathrm{CH}^r(X_1)$ be an algebraic cycle class. Then*

1. *the refined cycle class $c(\xi_1) \in \mathrm{H}^{2r}_{cont}(X_1, q(r)W_\bullet\Omega^\bullet) = \mathrm{H}^{2r}_{cont}(X_1, p(r)\Omega^\bullet_{X_\bullet})$ is of Hodge type iff and only if ξ_1 lies in the image of the restirction map $\mathrm{CH}^r_{cont}(X_\bullet) \to \mathrm{CH}^r(X_1)$.*

2. *the crystalline class $c_{cris}(\xi_1) \in \mathrm{H}^{2r}_{cont}(X_1, W_\bullet\Omega^\bullet) = \mathrm{H}^{2r}_{cont}(X_1, \Omega^\bullet_{X_\bullet})$ id of Hodge type modulo torsion if and only if $\xi \otimes \mathbb{Q}$ lies in the image of the restriction map $\mathrm{CH}^r_{cont}(X_\bullet) \otimes \mathbb{Q} \to \mathrm{CH}^r(X_1) \otimes \mathbb{Q}$.*

Proof: **Part 1**. The exact obstruction sequence (2) from above can be extended by the refined crystalline cycle class map to a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^r_{cont}(X_\bullet) \arrow[r] \arrow[d, "c"] & \mathrm{CH}^r(X_1) \arrow[r, "\mathrm{Ob}"] \arrow[d, "c"] & \mathrm{H}^{2r}_{cont}(X_1, p(r)\Omega^{<r}_{X_\bullet}) \arrow[d, equal] \\
\mathrm{H}^{2r}_{cont}(X_1, p(r)\Omega^{\geqslant r}_{X_\bullet}) \arrow[r] & \mathrm{H}^{2r}_{cont}(X_1, p(r)\Omega^\bullet_{X_\bullet}) \arrow[r] & \mathrm{H}^{2r}_{cont}(X_1, p(r)\Omega^{<r}_{X_\bullet})
\end{tikzcd}$$

with exact rows. It is clear by definition that the left square commutes. For the right square, recall that the morphism

$$\alpha : W_\bullet\Omega^r_{X_1,\log}[-r] \to p(r)\Omega^{<r}_{X_\bullet}$$

which induces the connecting morphism in the exact triangle is equal to the composition

$$\beta : W_\bullet\Omega^r_{X_1,\log}[-r] \to W_\bullet\Omega^{\geqslant r}_{X_1} \to q(r)W_\bullet\Omega^\bullet_{X_1} \to p(r)\Omega^\bullet_{X_\bullet} \to p(r)\Omega^{<r}_{X_\bullet}$$

which corresponds to the lower left path in the square in question. Then a diagram chase shows the claim.
Part 2. This follows directly from Part 1 if we bear in mind that $\mathrm{H}^2\, r_{cont}(X_1, p(r)\Omega^\bullet)$ and $\mathrm{H}^{2r}_{cont}(X_1, \Omega^\bullet_{X_\bullet})$ are rationally isomorphic. □
One question remains:

Question 4.2. Where in this discussion was it crucial that we restricted ourselves to X_1 being proper and how does one generalise this again then?

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