The motivic pro-complex

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1 Milnor K-theory

In this section we recall the definition and basic properties of Milnor K-theory for fields and rings. Following [17] we give a definition for the Milnor K-sheaves and state the Gersten conjecture in equi-characteristic.

1.1 Milnor K-theory for fields

We start by recalling the definition of the Milnor K-groups for fields in generators and relations along with some properties.

Let F be a field and $T^*(F)$ the tensor algebra of F. Let I be the two-sided homogenous ideal in $T^*(F)$ generated by the elements $a \otimes (1-a)$ with $a, 1-a \in F^*$.

Definition 1.1.1. The Milnor K-groups of the field F are defined to be

$$K_n^M(F) := T^n(F)/I.$$

They form a graded ring $K_*^M(F) = T^*(F)/I$. The class of $a_1 \otimes \cdots \otimes a_n$ in $K_n^M(F)$ is denoted by $\{a_1, \ldots, a_n\}$. Elements of I are usually called Steinberg relations.

The following basic properties are standard.

- $K_0^M(F) = \mathbb{Z}$, $K_1^M(F) = F^*$.
- For a field extension $F \hookrightarrow E$, $\Rightarrow$ there is a natural morphism $K_*^M(F) \to K_*^M(E)$.
- It is an **anticommutative** ring.
- For $a, a_i \in F^*$ with $a_1 + \cdot s a_n = 1$ or 0

$$\begin{aligned} \{a, -a\} &= \{a, -1\} \\ \{a_1, \ldots, a_n\} &= 0 \end{aligned}$$

1.2 The theory for local rings with infinite residue fields

We briefly recall Kerz's discussion of Milnor K-theory in the case when the residue fields have "enough" elements (see [17]).

Definition 1.2.1. For a regular semi-local ring R over a field k the Milnor K-groups are given by

$$K_n^M(R) = \operatorname{Ker}\left(\bigoplus_{x \in R^{(0)}} K_n^M(k(x)) \xrightarrow{\partial} \bigoplus_{y \in R^{(1)}} K_n^M(k(y)) \right).$$

In an attempt to generalise the definition of the Milnor K-ring for fields to arbitrary unital rings, one can define a graded ring in the following way:

Definition 1.2.2. For a unital ring R let

$$\overline{K}_*^M(R) = T^*(R)/J$$

where J is the two-sided homogeneous ideal generated by the Steinberg relations and elements of the form $a \otimes (-a)$.

If R is a regular semi-local ring over a field, there is a canonical homomorphism of groups

$$\overline{K}_i^M(R) \to K_i^M(R)$$

which is surjective if the base field is infinite (or sufficiently large, as in [17]). Kerz proves that in this case the additional relation $\{a, -a\} = 0$ in the definition is obsolete and that the usual relations hold.

We want to globalise this to schemes.

Definition 1.2.3. Define $\overline{\mathscr{K}}_*^M$ to be the Zariski sheaf associated to the presheaf

$$U \mapsto \overline{K}_*^M(\Gamma(U, \mathscr{O}_U))$$

on the category of schemes.

Inspired by Definition 1.2.1 one defines the following.

Definition 1.2.4. Let $\mathscr{K}_n^M$ be the sheaf

$$U \mapsto \operatorname{Ker}\left(\bigoplus_{x \in U^{(0)}} i_{x*}K_n^M(k(x)) \xrightarrow{\partial} \bigoplus_{y \in U^{(1)}} i_{y*}K_n^M(k(y))\right)$$

on the big Zariski site of regular varieties (schemes of finite type) over a field k, where i_x is the embedding of a point x in U.

One part of the Gersten conjecture for Milnor K-theory is to show that these two definitions coincide. Kato constructed a Gersten complex of Zariski sheaves for Milnor K-theory of a scheme X

$$0 \to \overline{\mathscr{K}}_n^M \to \bigoplus_{x \in X^{(0)}} i_{x*}K_n^M(k(x)) \to \bigoplus_{y \in X^{(1)}} i_{y*}K_n^M(k(y)) \to \cdots \tag{1}$$

In [23] Rost gives a proof that this sequence is exact if X is regular and of algebraic type over an arbitrary field k except possibly at the first two places. Exactness at the second place was shown independently by Gabber and Elbaz-Vincent/Müller-Stach. Finally Kerz proved that the Gersten complex is exact at the first place for X a regular scheme over a field, such that all residue fields are big enough.

In particular, this shows:

Corollary 1.2.5. *Let X be a regular scheme of dimension n over an infinite field. Then*

$$\mathscr{K}_*^M = \overline{\mathscr{K}}_*^M.$$

1.3 The theory for local rings with finite residue fields

As Kerz points out in [18], the Gersten conjecture does not hold in general if we use the same construction of Milnor K-theory for local rings with finite residue fields.

Let $\mathfrak{S}$ be the category of abelian sheaves on the big Zariski site of schemes and $\mathfrak{ST}$ the full subcategory of sheaves that admit a transfer (or norm) map in the sense of Kerz [18]. Furthermore, let $\mathfrak{ST}^\infty$ be the full subcategory of sheaves in $\mathfrak{S}$ which admit norms as described if we restrict the system to local A-algebras A' with infinite residue fields. An example would be the Milnor K-sheaf $\overline{\mathscr{K}}_n^M$ for every n.

A main result in Kerz's article [18] is that for a continuous functor $F \in \mathfrak{S}\mathfrak{T}^\infty$ there exists a continuous functor $\widehat{F} \in \mathfrak{S}\mathfrak{T}$ and a natural transformation satisfying a universal property. Namely, for an arbitrary continuous functor $G \in \mathfrak{S}\mathfrak{T}$ together with a natural transformation $F \to G$ there is a unique natural transformation $\widehat{F} \to G$ making the diagram

$$\begin{array}{ccc} F & \longrightarrow & \widehat{F} \\ & \searrow & \swarrow_{\exists !} \\ & G & \end{array}$$

commutative. Moreover, for a local ring with infinite residue field, the two functors coincide. It is constructed using rational function rings.

As a corollary we obtain an "improved" Milnor K-theory, taking into account that $\overline{\mathscr{K}}_n^M$ is in $\mathfrak{S}\mathfrak{T}^\infty$ and continuous.

Corollary 1.3.1. *For every $n \in \mathbb{N}$ there exists a universal continuous functor $\widehat{\mathscr{K}}_n^M \in \mathfrak{S}\mathfrak{T}$ and a natural transformation*

$$\overline{\mathscr{K}}_n^M \mapsto \widehat{\mathscr{K}}_n^M$$

such that for any continuous $G \in \mathfrak{S}\mathfrak{T}$ together with a natural transformation $\overline{\mathscr{K}}_n^M \to G$ there is a unique natural transformation $\widehat{\mathscr{K}}_n^M \to G$ such that the diagram

$$\begin{array}{ccc} \overline{\mathscr{K}}_n^M & \longrightarrow & \widehat{\mathscr{K}}_n^M \\ & \searrow & \swarrow_{\exists !} \\ & G & \end{array}$$

commutes.

In the affine case this is denoted by

$$K_*^M \mapsto \widehat{K}_*^M$$

We list some of the important properties, proved in [18, Proposition 10].

1. Let $(A, \mathfrak{m})$ be a local ring. Then $\widehat{K}_1^M(A) = A^\times$.
2. $\widehat{K}_*^M(A)$ has a natural structure as graded commutative ring.
3. The ring $\widehat{K}_*^M(A)$ is skew symmetric.
4. For $a_1, \ldots, a_n \in A^\times$ with $a_1 + \cdots + a_n = 1$ the image $\{a_1, \ldots, a_n\}$ of $a_1 \otimes \cdots \otimes a_n$ in $\widehat{K}_n^M(A)$ is trivial.
5. Let A be regular, equicharacteristic, F its quotient field and $X = \operatorname{Spec} A$. Then the Gersten conjecture holds, i.e. the Gersten complex

$$0 \to \widehat{K}_n^M(A) \to K_n^M(F) \to \oplus_{x \in X^{(1)}} K_{n-1}^M(k(x)) \to \cdots$$

In general, the natural map

$$\overline{\mathscr{K}}_*^M(X) \to \widehat{\mathscr{K}}_*^M(X)$$

is not an isomorphism. For example, the improved Milnor K-theory is equal to the Quillen K-theory for any local ring A, $\widehat{K}_2^M(A) = K_2(A)$, which is not true in this generality for the usual Milnor K-theory. An example for this was given by Bruno Kahn in the Appendix to [?]. However, from the fact that $\widehat{\mathscr{K}}_*^M$ satisfies the Gersten conjecture, we can deduce a useful corollary.

Corollary 1.3.2. *Let X be a smooth scheme with finite residue fields. Then*

$$\mathscr{K}_*^M = \widehat{\mathscr{K}}_*^M$$

where $\mathscr{K}_*^M$ is as in Definition 1.2.4.

Another important feature of the improved Milnor K-theory is that it is locally generated by symbols. In other words, it's elements satisfy the Steinberg relation. In fact Kerz shows the following theorem.

Theoreme 1.3.3. *Let A be a local ring. Then the map*

$$K_*^M(A) \to \widehat{K}_*^M(A)$$

is surjective.

Proof (idea): One can use the transfer map for extensions of local fields of degree 2 and 3 to reduce to the cases $n = 2$ and $n = 1$, whereof both are classical if one takes into account (1) of the list of properties above and that the improved Milnor K-theory is equal to the Quillen K-theory for any local ring A. □

1.4 Some deeper properties associated to the Milnor K-sheaf

Let $S = \operatorname{Spec} k$ for a perfect field k of positive characteristic p and X/k smooth. We know that the Milnor K-sheaf $\mathscr{K}_*^M$ on X is p-torsion free (Izhboldin or Geisser-Levine) and logarithmic differential map

$$d\log : \mathscr{K}^M / p^n \xrightarrow{\sim} W_n\Omega^r_{\log}$$

is an isomorphism (shown by Bloch-Kato or Geisser-Levine).

Let R be an essentially smooth local ring over $W_n(k)$ and set $R_n = R/p^n$. Over $\operatorname{Spec} R$ we consider the decreasing filtration of the Milnor K-ring

$$K_r^M(R) \supset U^1K_r^M(R) \supset U^2K_r^M(R) \supset \cdots \supset U^iK_r^M(R) \supset \cdots$$

where $U^iK_r^M(R)$ is generated by elements of the form $\{1 + p^ix, x_2, \ldots, x_r\}$ with $x \in R$ and $x_i \in R^*$. By definition $U^1K_r^M(R)$ is the kernel of the projection $K_r^M(R) \to K_r^M(R_1)$.

Lemma 1.4.1. *The groups $U^1K_r^M(R)$ is p-primary torsion of finite exponent.*

Proof: It is enough to show this for $r = 2$, where one can pass to relative K-groups. The calculation here is then easier. □

We will use the following theorem of Kurihara to relate Milnor K-theory and motivic cohomology of p-adic schemes.

Theoreme 1.4.2. *For $p > 2$ the map*

$$pxd\log y_1 \wedge \ldots \wedge d\log y_{r-1} \mapsto \{\exp(px), y_1, \ldots, y_{r-1}\}$$

induces an isomorphism

$$\operatorname{Exp} : p\Omega^{r-1}_{R_n}/p^2d\Omega^{r-2}_{R_n} \xrightarrow{\sim} U^1K_r^M(R_n).$$

Proof: This is done in three steps. One first shows that the exponential map is well-defined on $p\Omega^{r-1}_{R_n}$. Then that it factors through the quotient. The last part is to show that it is an isomorphism.

1st step. Kurihara shows that the morphism is well defined if $K_r^M(R)$ is replaced by its p-adic completion. As above, it is sufficient to show the claim for $r = 2$. As mentioned before, $K_2^M(R_1)$ is p-tprsion free. Thus for any n

$$0 \to U^1K_2^M(R) \otimes \mathbb{Z}/p^n \to K_2^M(R) \otimes \mathbb{Z}/p^n \to K_2^M(R_1) \otimes \mathbb{Z}/p^n \to 0$$

is exact. For n large enough the lemma says that $U^1K_2^M(R)\otimes\mathbb{Z}/p^n \cong U^1K_2^M(R)$. Taking inverse limits in the exact sequence, we get that

$$U^1K_2^M(R)\to \widehat{K_2^M(R)} \tag{2}$$

is injective, and we obtain the claim from Kurihara's result.

2^nd^ step. To show that the morphism factors through the quotient, we show that $\mathrm{Exp}(p^2d\Omega_R^{r-2})=0$. Again wlog $r=2$. We use again the injectivity of (2) and the fact that the claim has been shown for $\widehat{K_2^M(R)}$ by Kurihara.

3^rd^ step. To show that the exponential map on the quotient is an isomorphism, set $G_r=p\Omega_R^{r-1}/\Omega_R^{r-2}$ and define a filtration by

$$U^iG_r=p^i\Omega_R^{r-1}/\Omega_R^{r-2}.$$

Kurihara shows that the graded pieces of this filtration are isomorphic to the graded pieces of $K_r^M(R)$. Therefore, the exponential map is an isomorphism. □

The next interesting result is the relationship between Milnor K-theory and the motivic complex (resp. motivic cohomology). In fact it is now known that

$$\mathscr{K}_n^M=\mathscr{H}^n(\mathbb{Z}(n))$$

where the lefthand side is the Milnor K-sheaf and the righthand side is the motivic cohomology sheaf. This is sometimes called Beilinson's conjecture. It was shown by Kerz in [17]. The idea of the proof is as follows.

In [24] Suslin and Voevodsky show that the claim is true for a field F. Recall that $\mathrm{H}^{n,n}(X,\mathbb{Z}):=\mathrm{H}^n_{mot}(X,\mathbb{Z}(n))=\mathbb{H}^n_{Nis}(X,\mathbb{Z}(n))$ where

$$\mathbb{Z}(q)=C_*\,\mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge q})[-q]$$

is a presheaf with transfers, obtained via a simplicial complex. Every n-tuple $(a_1,\dots,a_n)$ of elements in the base field F defines an F-rational point $(a_1,\dots,a_n)\in\mathbb{G}_m^n$. The class of it in $\mathrm{H}^{n,n}(F,\mathbb{Z})$ is denoted by $[a_1,\dots,a_n]$. One shows that elements of the form $(a,1-a)$ are mapped to zero, so this defines a morphism

$$K_n^M(F)\to\mathrm{H}^{n,n}(F,\mathbb{Z}).$$

Suslin Voevodsky show that this is surjective and construct an iverse.

Furthermore, it is well-known, that motivic cohomology satisfies the Gersten conjecture. Kerz on the other hand shows, that the Milnor K-sheaf as well satisfies the Gersten conjecture. This leads to a commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathscr{K}_n^M\big|_X & \longrightarrow & \oplus_{x\in X^{(0)}}K_n^M(x) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{H}^n(\mathbb{Z}(n))\big|_X & \longrightarrow & \oplus_{x\in X^{(0)}}\mathrm{H}^n(x,\mathbb{Z}(n)) & \longrightarrow & \cdots
\end{array}$$

Since we have isomorphisms on the field level and both lines are exact, this shows, that the first vertical map is an isomorphism as well.

A similar reasoning leads to a Bloch formula

$$\mathrm{H}^n(X,\mathscr{K}_n^M)\cong\mathrm{CH}^n(X)$$

if X is refular, contains a field (Kerz states it for infinite residue fields, but with his improved Milnor K-theory it should be true also for finite residue fields).

2 The motivic procomplex

2.1 Definition and basic properties

Recall from two weeks ago the definition of motivic cohomology and the motivic complex. For X/k smooth, let $\mathbb{Z}_{tr}(X) = \mathscr{C}or(-, X)$. This is a presheaf with transfers. The motivic complex

$$M(X) := C_* \mathbb{Z}_{tr}(X)$$

is the complex assiciated to the simplicial presheaf given by $U \mapsto \mathbb{Z}_{tr}(X \times \Delta^\bullet)$. We then define the Suslin-Voevodsky complex by

$$\mathbb{Z}(r) := C_* \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge r}[-r].$$

In sum one has

$$\mathbb{Z}(r)^i(U) = \mathscr{C}or(U \times_k \Delta^{r-i}, \mathbb{G}_m^{\wedge r}).$$

It is supported in degree $\leqslant r$. For a smooth scheme over k, $\mathbb{Z}_X(r)$ denotes the restriction of $\mathbb{Z}(r)$ to the small Nisnevich site of X.

Notation 2.1.1. Recall the notation from earlier sections. Let k be a perfect field of characteristic $p > 0$ and $W = W(k)$ the ring of Witt vectors which is an adic ring with ideal of definition $I = (p)$. Let $X_\bullet \in \mathrm{Sch}_{W_\bullet}$ (a p-adic formal scheme over the Witt vectors). We denote $X_n = X_\bullet \otimes W_n(k)$. Then in particular $X_1 = W \otimes_W k$ is its special fiber.

Forthermore recall $\mathfrak{S}_{X_\bullet}(r) = \mathrm{cone}(J(r)\Omega^\bullet_{D_\bullet} \xrightarrow{1-f_r} \Omega^\bullet_{D_\bullet}[-1]$ is the syntomic complex of Fontaine-Messing.

We will $\mathbb{Z}_{X_1}(r)$ consider both as an object in the derived category $D(X_1) = D(X_1)_{Nis}$ And as a constant pro-complex in $D_{pro}(X_1) = D_{pro}(X_1)_{Nis}$. Using the euality

$$\mathscr{H}^r(\mathbb{Z}(r)) = \mathscr{K}^M_r$$

we define a logarithm map

$$d\log : \mathbb{Z}_{X_1}(r) \to \mathscr{H}(\mathbb{Z}_{X_1}(r))[-r] = \mathscr{K}^M_{X_1,r}[-r] \xrightarrow{d\log} W_\bullet\Omega^r_{X_1,\log}[-r]$$

in $D_{pro}(X_1)$. The second part of the map is an isomorphism, since the logarithmic differentials are generated by symbols. Recall that we have a map

$$\Phi^J : \mathfrak{S}_{X_\bullet}(r) \to W_\bullet\Omega^r_{X_1,\log}[-r]$$

in $D_{pro}(X_1)$ that fits into an exact fundamental triangle.

Definition 2.1.2. Assume $p > r$. We define the motivic procomplex by

$$\mathbb{Z}_{X_\bullet}(r) = \mathrm{cone}(\mathfrak{S}_{X_\bullet}(r) \oplus \mathbb{Z}_{X_1}(r) \xrightarrow{\Phi^J \oplus (-\log)} W_\bullet\Omega^r_{X_1,\log}[-r])[-1]$$

as object of $D_{pro}(X_1)$.

After Beilinson-Bernstein-Deligne the cone is well defined up to unique isomorphism. We will prove some properties of the motivic pro-complex.

Proposition 2.1.3. *1. $\mathbb{Z}_{X_\bullet}(0) = \mathbb{Z}$ is the constant sheaf in degree zero.*

2. One has $\mathbb{Z}_{X_\bullet}(1) = \mathbb{G}_{m,X_\bullet}[-1]$.

3. The pro-complex $\mathbb{Z}_{X_\bullet}(r)$ is supported in cohomological degrees $\leqslant r$.

4. One has $\mathbb{Z}_{X_\bullet}(r) \otimes^L_{\mathbb{Z}} \mathbb{Z}/p^\bullet = \mathfrak{S}_{X_\bullet}(r)$ in $D_{pro}(X_1)$.

5. *There is a Beilinson type formula* $\mathcal{H}^r(r)) = \mathscr{K}^M_{X_\bullet,r}$ *in* $\mathrm{Sh}_{pro}(X_1)$.
6. *There is a canonical product structure*

$$\mathbb{Z}_{X_\bullet}(r) \otimes^L_{\mathbb{Z}} \mathbb{Z}_{X_\bullet}(r') \to \mathbb{Z}_{X_\bullet}(r+r')$$

compatible with the product on the usual motivic complex over X_1 *and on the syntomic complex.*

PROOF: To show **(1)**, one has $W_\bullet\Omega^0_{X_1,\log} = \mathbb{Z}/p^\bullet$, $\mathbb{Z}_{X_1}(0) = \mathbb{Z}$ and $\mathfrak{S}_{X_\bullet}(0) = \mathbb{Z}/p^\bullet$. So the statement follows directly from the definition.

We show **(3)**. There is a long exact sequence

$$\ldots \to \mathcal{H}^i(\mathbb{Z}_{X_\bullet}(r)) \to \mathcal{H}^i(\mathfrak{S}_{X_\bullet}(r)) \oplus \mathcal{H}^i(\mathbb{Z}_{X_1}(r)) \to \mathcal{H}^i(W_\bullet\Omega^r_{X_1,\log}[-r]) \to \ldots$$

where the second map is induced by $\Phi^J \oplus (-\log)$. We have seen earlier that $\mathfrak{S}_{X_\bullet}(r)$ has support in $[1,r]$. (Beilinson-Soulé predicts the same for the motivic complex.) But as the $d\log$-mao is an epimorphism, this shows the claimed support for the motivic pro-complex.

To show **(5)**. For $i = r$ we have a short exact sequence

$$0 \to \mathcal{H}^r(\mathbb{Z}_{X_\bullet}(r)) \to \mathcal{H}^r(\mathfrak{S}_{X_\bullet}(r)) \oplus \mathcal{H}^r(\mathbb{Z}_{X_1}(r)) \xrightarrow{\Phi^J\oplus(-\log)} W_\bullet\Omega^r_{X_1,\log} \to 0$$

The exact fundamental triangle from Theorem 4.4 gives an exact sequence

$$0 \to p\Omega^{r-1}_{X_\bullet}/p^2 d\Omega^{r-2}_{X_\bullet} \to \mathcal{H}^r(\mathfrak{S}_{X_\bullet}(r)) \xrightarrow{\Phi^J} W_\bullet\Omega^r_{X_1,\log} \to 0$$

These two sequences induce a third exact sequence, which can be put into a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & p\Omega^{r-1}_{X_\bullet}/p^2 d\Omega^{r-2}_{X_\bullet} & \longrightarrow & \mathcal{H}^r(\mathbb{Z}_{X_\bullet}(r)) & \longrightarrow & \mathcal{H}^r(\mathbb{Z}_{X_1}(r)) & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow \wr & & \\
0 & \longrightarrow & p\Omega^{r-1}_{X_\bullet}/p^2 d\Omega^{r-2}_{X_\bullet} & \longrightarrow & \mathscr{K}^M_{X_\bullet,r} & \longrightarrow & \mathscr{K}^M_{X_1,r} & \longrightarrow & 0
\end{array}$$

which is induced by the exponential map for the Milnor K-sheaf which we talked about earlier. The middle vertical map is Kato's regulator map.

For **(2)**. The Beilinson-Soulé vanishing is clear for $r = 1$. So from (3) which tells us about the cohomological support and (5) one obtains the formula in (2).

For **(3)**. As $W_n\Omega^r_{X_1,\log}$ is a flat $\mathbb{Z}/p^n$ module,

$$W_\bullet\Omega^r_{X_1,\log} \otimes^L_{\mathbb{Z}} \mathbb{Z}/p^\bullet = W_\bullet\Omega^r_{X_1,\log}$$

in $D_{pro}(X_1)$. By the fundamental triangel of Theorem 4.4 the same is true for the syntomic complex. By Geisser-Levine

$$\mathbb{Z}_{X_1}(r) \otimes^L_{\mathbb{Z}} \mathbb{Z}/p^n = W_n\Omega^r_{X_1,\log}[-r]$$

so $\mathbb{Z}_{X_\bullet}(r) \otimes^L_{\mathbb{Z}} \mathbb{Z}/p^\bullet = \mathfrak{S}_{X_\bullet}(r)$ in $D_{pro}(X_1)$.

For **(6)**. The product structure follows from the product structure of the syntomic nad regular motivic complexes. □

2.2 The motivic fundamental triangle

Now we come to the motivic fundamental triangle.

Proposition 2.2.1. *One has a unique commutative diagram of exact triangles in $D_{pro}(X_1)$*

$$\begin{array}{ccccccc}
p(r)\Omega^{\leq r}_{X_\bullet}[-1] & \longrightarrow & \mathbb{Z}_{X_\bullet}(r) & \longrightarrow & \mathbb{Z}_{X_1}(r) & \longrightarrow & \cdots \\
\| & & \downarrow & & & & \\
p(r)\Omega^{\leq r}_{X_\bullet}[-1] & \longrightarrow & \mathfrak{S}_{X_\bullet}(r) & \longrightarrow & W_\bullet\Omega^r_{X_1,\log}[-r] & \longrightarrow & \cdots
\end{array}$$

where the bottom comes from the fundamental triangle from Theorem 4.4 and the maps in the right square are the canonical ones.

Proof: As the right square consists of the canonical maps, it is homotopy cartesian by definition. The existence of the commutative diagram is then a stnadard result about triangulated categories by Neeman.

For the uniqueness, one has to show that the first morhiism in the upper row

$$p(r)\Omega^{\leqslant r-1}_{X_\bullet}[-1] \to \mathbb{Z}_{X_\bullet}(r)$$

is uniquely defined by the conditions of the proposition. This is also a standard result in triangulated categories by Beilinson-Bernstein-Deligne. □

Corollary 2.2.2. *For $Y_\bullet = X_\bullet \times \mathbb{P}^m$ one has a projective bundle formula:*

$$\bigoplus_{s=0}^{m} \mathrm{H}^{r'-2s}_{cont}(X_1, \mathbb{Z}_{X_\bullet}(r-s)) \to \mathrm{H}^{r'}_{cont}(Y_1, \mathbb{Z}_{Y_\bullet}(r))$$

is an isomorphism.

Proof: By the previous proposition one has to show the formula for the Suslin-Voevodsky motivic cohomology and for the Hodge cohomology. This has been done in [19] and Deligne/Grothendieck in SGA7 respectively. □

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