Seminar on p-adic deformation of algebraic cycle classes after S. Bloch, H. Esnault and M.Kerz

Student Number Theory Seminar
University of Utah

27th January 2013

1 Introduction

The goal of this seminar is to read the article [5]. By studying the p-adic deformation properties of algebraic cycles (modulo rational equivalences), they show that the crystalline Chern character of a vector bundle lies in a certain part of the Hodge filtration if and only if the vector bundle lifts to a formal pro-class in K-theory on the p-adic scheme. This is part of the p-adic analogue of Grothendieck's variational Hodge conjecturs [12, Note 13].

2 Suggested planning for the talks

These are just some suggestions.

2.1 Introduction (23rd Jan. 2013)

Follow the introduction of [5].

- The variational Hodge conjecture [12, Note 13], [5, Conj.1.1].
- How this can be deduce from the Hodge conjecture.
- The p-adic variational Hodge conjecture [9], [8], [5, Conj.1.1].
- The case of line bundles, strategy of [5].
- Main result [5, Theorem 1.3].

2.2 The de Rham and de Rham-Witt complex (30th Jan. 2013)

Technical properties of the de Rham and de Rham-Witt complexes [5, Section 2].

- Recall the etale and Nisnevich sites [19], and $D_{\mathrm{pro}}(X_1)_{\text{ét},\mathrm{Nis}}$ – the category of pro-systems in the derived category (see also [5, Appendix A,B]).
- Definition of subcomplexes $p(r)\Omega^*_{X_\bullet} \subset \Omega^*_{X_\bullet}$, $q(r)W_\bullet\Omega^*_{X_1} \subset W_\bullet\Omega^*_{X_1}$ and of the logarithmic subsheaves $W_\bullet\Omega^r_{X_1,\log} \subset W_\bullet\Omega^r_{X_1}$.
- Comparison isomorphisms in $D_{\mathrm{pro}}(X_1)_{\mathrm{Nis}}$: $\Omega^*_{X_\bullet} \xrightarrow{\sim} W_\bullet\Omega^*_{X_1}$ and $p(r)\Omega^*_{X_\bullet} \xrightarrow{\sim} q(r)W_\bullet\Omega^*_{X_1}$.

2.3 The syntomic complex $\mathfrak{S}_{X_\bullet}(r)$ of FOntaine-Messing (6th Feb. and 13th Feb. 2013)

Cover Sections 3, 4 and 5 of [5].

- The syntomic (étale and Nisnevich) complex – [5, Def. 3.2].
- Distinguished triangle for $\mathfrak{S}_{X_\bullet}(r)$ ([5, Theo. 4.4]).
- Description of the connecting morphism ([5, Theo. 5.1]).

2.4 The motivic complex of Suslin-Voevodsky (20th Feb 2013)

The definition of the motivic complex $\mathbb{Z}(r)$ (existence conjectured by Beilinson and Lichtenbaum) on the big (Zariski or Nisnevich) site. In preparation for the construction of the motivic procomplex, we consider also the restriction $\mathbb{Z}_{X_1}(r)$ of $\mathbb{Z}(r)$ to the smal Nisnevich site of X_1.

- Motivations - axiomatic conditions, comparison results (e.g. [10, Introduction] or [17, Preface]).
- Category of finite correspondences, presheaves with transfers, simplicial presheaves [17, Lect. 1 and 2].
- Formal definition of $\mathbb{Z}(r)$ and $\mathbb{Z}_X$. Elementary properties [17, Lect. 3] [20, Section 3].

2.5 Milnor K-theory and Chow groups (27th Feb. 2013)

The connections between $\mathbb{Z}(r)$, Milnor K-theory and Chow groups.

- Milnor K-groups and K-sheaves (see also [5, Section 11]).
- Milnor K-theory and the motivic complex ([20, Theo. 3.4], [17, Lect. 5], [16, Theo. 7.6], [5, (6.3)]).
- Chow groups and the motivic complex ([16, Theo. 7.5], [17, Lect. 19]).

2.6 The motivic pro-complex $\mathbb{Z}_{X_\bullet}(r)$ (6th Mar. 2013)

Cover Section 6 of [5].

- Definition of the motivic pro-complex.
- Basic properties.
- Motivic fundamental triangle.

2.7 Continuous cohomology (20th Mar. 2013)

Introduce continuous cohomology after Jannsen (the case of projective systems of sheavs on the étale site of a scheme) [15]. The presentation in [5, Appendix B] is a little different.

- Motivations.
- Definition [15, Sections 1,2,3].
- Cup product and Chern classes [15, Sections 3 and 6]. Jannsen treats the ℓ-adic continuous cohomology, whereas we will need this formalism for crystalline cohomology.

2.8 Crystalline Hodge Obstruction (27th Mar. 2013)

Cover Section 7 of [5]. One of the main ingredients in the proof of the main result.

- Recall: comparison between crystalline cohomology and continuous de Rham-Witt cohomology.
- Continuous Chow groups.
- Construction of crystalline cycle class maps and refined crystalline cycle classes.
- Hodge classes (Def. 7.3) and connection to initial definition in [5, Conj. 1.2].
- First main result: Theorem 7.4.

2.9 Continuous K-theory and Chern classes (3rd April 2013)

Cover [5, Section 8]. Quillen's +- and Q-construction for the K-theory of p-adic formal schemes. Universal Chern classes.

- Continuous K-theory.
- Construction of univesal classes by Gillet.

2.10 Results from topological cyclic homology (10th April 2013)

Cover [5, Section 9]. Results due to McCarthy, Madsen, Hesselholt, Geisser,...

2.11 Chern character isomorphism (17th April 2013)

Cover [5, Section 10]. Second ingredient to the main result.

- Proof of Theorem 10.1.
- Combine Theorems 7.5 and 10.1 to obtain main result.

2.12 Variational p-adic Hodge conjecture for line bundles. (24th April 2013)

If there is time left at the end. Proof of [4, Theorem 3.8]. See as well [7].

References

[1] GROTHENDIECK A., DIEUDONNÉ J.: *Eléments de géométrie algébrique III: étude cohomologique des faisceaux cohérents.*

[2] BERTHELOT P.: *Cohomologie cristalline des schémas de caractéristique* $p > 0$. Lecture Notes in Mathematics **127**, Springer-Verlag, (1974).

[3] BERTHELOT P., OGUS A.: *Notes on crystalline cohomology.* Math.Notes **21**, Princeton University Press, (1978).

[4] BERTHELOT P., OGUS A.: *F-isocrystals and the de Rham cohomology I.* Inv.Math. **72**, 159-199, (1983).

[5] BLOCH S., ESNAULT H., KERZ M.: *p-adic deformations of algebraic cycle classes.* Preprint 2012, ArXiv:1203.2776v1.

[6] CHAMBERT-LOIR A.: *Cohomologie cristalline: un survol.* Exp.Math. **16**, 336-382, (1998).

[7] DELIGNE P.: *Cristaux ordinaires et coordonnées canoniques.* In algebraic Surfaces (Orsay 1976/78), L.N.M. **868**, 80-137, Springer-Verlag, (1981).

[8] EMERTON M.: *A p-adic variational Hodge conjecture and modular forms with multiplication.* Preprint 2012.

[9] FONTAINE J.-M., MESSING W.: *p-adic periods and étale cohomology.* Contemporary Math **87**, 176-207, (1987).

[10] FRIEDLANDER E.M., SUSLIN A., VOEVODSKY V.: *Cycles, transfers and motivic homology theories.* Annals of math. Studies **143**, Princeton University Press, (2000).

[11] GROS M.: *Classes de Chern et classes de cycles en cohomologie de Hodge-Witt logarithmique.* Mémoires de la S.M.F. 2e série, tome 21, 1-87, (1985).

[12] GROTHENDIECK A.: *On the de Rham cohomology of algebraic varieties.* Publ. math. I.H.E.S. **29**, 95-103, (1966).

[13] HARTSHORNE R.: *Algebraic Geometry.* Graduate Texts in Mathematics **52**, Springer-Verlag, (1977).

[14] ILLUSIE L.: *Grothendieck's existence theorem in formal geometry.* In Grothendieck's Fga explained, B. Fantechi et al. eds, M.S.M. **123**, 179-234, (2005).

[15] JANNSEN U.: *Continuous étale cohomology.* Math. Ann. **280**, 207-245, (1988).

[16] KERZ M.: *The Gersten conjecture for Milnor K-theory.* Invent. Math. 175, 1-33,(2009).

[17] MAZZA C., VOEVODSKY V., WEIBEL V.: *Lecture Notes on Motivic Cohomology.* Clay Mathematics Monographs **2**, A.M.s., (2006).

[18] MILNE J.S.: *Etale cohomology.* Princeton Mathematical Series **33**, Princton University Press, (1980).

[19] NISNEVICH Y.A.: *The completely decomposed topology on schemes and associated descent spectral sequences in algebraic K-theory.* In J.F. Jardine and V.P. Snaith. Algebraic K-theory: connections with geometry and topology. Proceedings of the NATO Advanced Study Institute held in Lake Louise, Alberta, December 7–11, 1987. NATO Advanced Science Institutes Series, C **279**. Dordrecht: Kluwer Academic Publishers Group, 241-342, (1989).

[20] SUSLIN A., VOEVODSKY V.: *Bloch-Kato conjecture and motivic cohomology with finite coefficients.* In The Arithmetic and Geometry of Algebraic Cycles, Nato Science Series, C **548**, 117-189, (2002).

[21] WEIBEL V.: *An introduction to homological algebra.* Cambridge Studies in Advanced Mathematics **38**, Cambridge University Press, (1994).