The admissible topology

Graduate Student Seminar in Arithmetic Geometry
Universität Regensburg

Winter term 2014/15

In this talk we introduce domains that generate a Grothendieck topology on affinoid spaces, and are the building blocks of rigid spaces as we will see later. It is based on Section 2.2 of [1]

1 Affinoid subdomains

We have already seen examples of affinoid subdomains, and want now to make a precise definition.

Definition 1.1. Let A be a k-affinoid algebra. A subset $U \subset M(A)$ is an affinoid subdomain, if there is a map

$$i : A \to A'$$

of k-affinoid algebras, such that the induced map

$$M(i) : M(A') \to M(A)$$

has image in U and is universal for this property.

Universality in the above definition means, that if we consider another map $\varphi : A \to B$ of affinoid algebras, there is a commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[dr, "\varphi"'] & A' \arrow[d, "\exists"] \\ & B \end{tikzcd}$$

if and only if the induced map $M(\varphi)$ has image in U, i.e. the dual diagram is

$$\begin{tikzcd} M(A') \arrow[r] & U \arrow[r, hook] & M(A) \\ M(B) \arrow[u] \arrow[ur] & & \end{tikzcd}$$

In this case, the diagram is unique.

If U is an affinoid subdomain, then it determines the associated affinoid algebra uniquely (up to unique A-algebra isomorphism), thus we may write A_U for A' in the definition. For obvious analogies, we call A_U the coordinate ring of U. And in fact one can identify $U = M(A_U)$.

Lemma 1.2. *The induced map*

$$M(A_U) \to M(A)$$

is a bijection on points in U.

Proof. First we show injectivity. Let x_1 and x_2 be two points in $M(A_U)$ that map to the same point $y \in U$. Thus there is a commutative diagram

$$\begin{tikzcd}
A \arrow[rr] \arrow[drr] \arrow[ddr, dashed] \arrow[ddrrr, dashed] & & A_U \arrow[d] & \\
& & A/\mathfrak{m}_y \arrow[dl] \arrow[dr] & \\
& A_U/\mathfrak{m}_{x_1} & & A_U/\mathfrak{m}_{x_2}
\end{tikzcd}$$

where $\mathfrak{m}_{x_i}$ are the maximal ideals corresponding to the points x_i. By the universal property (more precisely the uniqueness), it follows that x_1 and x_2 are identified.

On the other hand, surjectivity: Let $x \in U \subset M(A)$. Then there is a morphism $\phi : A \to A/\mathfrak{m}_x = \kappa(x) \to \overline{k}$, and the induced dual morphism is $M(\phi) : M(\kappa(x)) \to U \subset M(A)$. By the universal property, there is a map $A_U \to \kappa(x)$ such that

$$\begin{tikzcd}
A \arrow[r] \arrow[dr] & A_U \arrow[d, dashed, "\exists \phi_U"] \\
& \kappa x
\end{tikzcd}$$

commutes. The map $\phi_U : A_U \to \kappa(x) \to \overline{k}$ is an element in $A_U(\overline{k}) / \operatorname{Aut}(\overline{k}/k) \cong M(A_U)$, thus x can be interpreted as an element in $M(A_U)$. $\square$

Similarly, one can prove the claim of the following lemma.

Lemma 1.3. *If $V \subset U \subset M(A)$ are affinoid subdomains, then there is a unique A-algebra map*

$$\rho_V^U : A_U \to A_V.$$

Moreover, this construction is transitive with respect to another inclusion.

Hence, one can say that this pals the role of a restriction map.

Similar to scheme theory, one can show

Lemma 1.4. *For any affinoid subdomain, A_U is A-flat.*

Examples 1.5. By definition, Weierstraß, Laurent and rational domains are affinoid subdomains, and moreover they are open for the canonical topology. However, if one changes the definition of a rational domain slightly, and drop the condition, “no common zero”, the result is not open in the canonical topology any more, but still affinoid.

In order to copy more parts of scheme theory, we will need an appropriate notion of tensor products.

Let A and A' be k-affinoid algebras. The completed tensor product $A \widehat{\otimes}_k A'$ is defined as follows. Choose presentations $A \cong T_n/I$ and $A' \cong T_{n'}/I'$. Then via the natural maps $T_n \to T_{n+n'}$ and $T_{n'} \to T_{n+n'}$ respectively, we may consider the ideals $J, J' \subset T_{n+n'}$ generated by I nd I'. The k-affinoid algebra $T_{n+n'}/(J + J')$ comes with maps $\iota : A \to T_{n+n'}/(J + J')$ and $\iota' : A' \to T_{n+n'}/(J + J')$. This satisfies a universal property and the triple $(T_{n+n'}/(J + J'), \iota, \iota')$ is unique up to unique isomorphism. Consequently, we can denote this by $A \widehat{\otimes}_k A'$.

More generally, for a k-affinoid algebra A'' and maps $j : A'' \to A$ and $j' : A'' \to A'$ of k-affinoid algebras, one sets

$$A \widehat{\otimes}_{A''} A' = \left(A \widehat{\otimes}_k A'\right) / \left(j(a'') \widehat{\otimes} 1 - 1 \widehat{\otimes} j'(A'') \,\middle|\, a'' \in A''\right)$$

which also satisfies a universal property as expected.

For an analytic extension K/k one also can define a K-affinoid algebra

$$K \widehat{\otimes}_k A$$

However, base extension does not behave so well in rigid analytic geometry, as one would hope.

The completed tensor product from above, helps us to define intersections of affinoids as another affinoid domain. For two affinoid subdomains $U, U' \subset M(A)$ consider the tensor product

$$A_U \widehat{\otimes}_A A_{U'}$$

The universal property of the tensor product corresponds exactly to the universal property of the domain $U \cap U'$ which shows that this is in fact affinoid with

$$A_{U \cap U'} = A_U \widehat{\otimes}_A A_{U'}.$$

Similarly, one can define base change: let $\phi : A \to B$ be a map of k-affinoid algebras, and $U \subset M(A)$ be affinoid. Then $M(\phi)^{-1}(U) \subset M(B)$ is affinoid, with coordinate ring $A_U \widehat{\otimes}_A B$.

Although the introduction of affinoid subdomains are crucial for the theory, they can seem difficult to handle because of the abstract definition. However, by the Gerritzen-Grauert theorem, one can describe them in terms of rational domains, which are fairly explicit.

Theorem 1.6. *Let A be k-affinoid. Every affinoid subdomain $U \subset M(A)$ is a finite union of rational domains.*

In particular, they are open with respect to the canonical topology.

It is still very hard to see wether an arbitrary union of rational domains is affinoid. Moreover, in order to get a good theory, one cannot allow arbitrary unions of affinoid domains. Tate's idea was to restrict the class of open subsets, and also the allowed coverings, to force the topology to behave well. For example, affinoids should behave as if they were compact. In order to achieve this, we have to impose finiteness conditions.

Definition 1.7. Let A be a k-affinoid algebra. A subset $U \subset M(A)$ is an admissible open, if it can be covered (set-theoretically) by affinoid subdomains $U_i \subset M(A)$ such that for any map of k-affinoid algebras

$$\phi : A \to B$$

with the dual map

$$M(\phi) : M(B) \to M(A)$$

with image in U, this image can be covered by finitely many U_i's.

This is equivalent to say, that the open covering $\{M(\phi)^{-1}(U_i)\}$ of $M(B)$ has a finite subcovering. This not as strict as to say that the covering $\{U_i\}$ of U has to have a finite subcover.

Definition 1.8. A collection $\{V_i\}$ of admissible open subsets of $M(A)$ is an admissible cover of its union V, if for any map of k-affinoid algebras $\phi : A \to B$ such that $M(\phi)$ has image in V the covering $\{M(\phi)^{-1}(V_i)\}$ of $M(B)$ has a refinement by a covering consisting of finitely many affinoid subdomains.

It follows, that V is as well admissible open. On the other hand, the set-theoretic covering $\{U_i\}$ in the definition of admissible open, is also an admissible covering. a trivial example are finite covers of their union.

Now it can be seen, that this definition of "open" sets and "coverings" fix the problem of disconnectednes. One pathology occurred with the unit disk $M(T_n)$, where in the canonical topology, there is a cover, consisting of the "boundary" $V = \{|\underline{x}| = 1\}$ and the subset $U = \{|\underline{x}| < 1\}$ which is open for the canonical topology, and moreover an admissible open (covered by the domains $U_n = \{|\underline{x}| \leqslant \rho^{\frac{1}{n}}\}$ and this covering satisfies the finiteness condition of admissibility, since we only have to have finiteness for any affinoids that are contained in U). It follows, that V and U cover $M(T_n)$ set-theoretically and in the canonical topology, and U and V are even admissible – but disjoint!

This should be fixed in the admissible topology, it was designed to ban such situations. And indeed, if the covering $\{U, V\}$ was admissible, it would follow, that it has a refinement a finite covering of $M(T_n)$ by affinoids. By the Maximum Modulus Principle, any affinoid subdomain of $M(T_n)$ (that is not $M(T_n)$ itself) is contained in some U_n und as such in U. Thus any finite refinement of $\{U, V\}$ would produce a cover consisting of V and U_{n_0} for n_0 big enough. But one can easily find a point in $M(T_n)$ that is not contained in $V \cup U_{n_0}$.

References

[1] Brian Conrad. Several approaches to non-archimedean geometry. In *p-adic Geometry*, volume 78 of *AMS University Lecture Series*. Amer. Math. Soc., Providence, RI 41, 2008. Lectures from the 2007 Arizona Winter School.