5 The big de Rham–Witt complex

In this section we will introduce the big de Rham–Witt complex following Lars Hesselholt's paper [4] in Section 4. The original definition is due to Hesselholt and Madsen in [5] which relies on the adjoint functor theorem. However, there was an issue with 2-torsion. This was solved by Lars Hesselholt using λ-ring theory.

We will see how this construction generalises the p-typical de Rham–Witt complex from $\mathbb{F}_p$-algebras to $\mathbb{Z}_{(p)}$-algebras. At the end, we want to draw the relation to K-theory.

5.1 Big Witt complexes

Let S be a truncation set (recall that a truncation set is a subset $S \subset \mathbb{N}$ such that if $n \in S$ and $d|n$ then also $d \in S$). We will define the de Rham–Witt complex $\mathbb{W}\,\Omega_S$.

Let $\mathscr{J}$ be the set of truncation sets, partially ordered for inclusion. We consider it as a category with a morphism from T to S if $T \subset S$.It is clear that the assignment

$$S \mapsto \frac{S}{n}$$

is an endofunctor of $\mathscr{J}$. And since $\frac{S}{n} \subset S$ there is a morphism from $\frac{S}{n}$ to S.

Recall that we defined a ring functor for each truncation set S

$$A \mapsto \mathbb{W}_S(A),$$

called the big Witt vectors. Now, instead of fixing S, we fix a ring A to get a contravariant functor

$$\begin{array}{rcl} \mathscr{J} & \to & \mathcal{Ann} \\ S & \mapsto & \mathbb{W}_S(A) \end{array}$$

from $\mathscr{J}$ to the category of rings, sending colimits to limits. Recall that we defined Frobenius and Verschiebung for any $n \in \mathbb{N}$

$$\begin{array}{rcl} F_n : \mathbb{W}_S(A) & \to & \mathbb{W}_{\frac{S}{n}}(A) \\ V_n : \mathbb{W}_{\frac{S}{n}}(A) & \to & \mathbb{W}_S(A) \end{array}$$

where the former is a ring homomorphism and the latter is additive (a morphism of abelian groups). These deine in fact natural transformations with respect to the "variable" S.

We will now consider the category of big Witt complexes. The de Rham-Witt complex for a truncation set S can then be defined as the initial object in this category.

Remark 5.1. This is reminiscent of the category of de Rham-V-procomplexes, whose initial object was the p-typical de Rham–Witt complex. One difference is, that here we need from the beginning a Frobenius, whereas in the p-typical case, the Frobenius came out of an explicit construction after having established the existence of an initial object. It should be remarked however, that in the case of the p-typical de Rham–Witt complex, one can also adopt a similar approach. In fact, there is a forgetful functor from the category of de Rham-V-procomplexes to the category of Witt complexes, simply forgetting the Frobenius. The de Rham–Witt complex can be defined as the initial object in either of them.

As mentioned above, the original definition of big Witt complexes due to Hesselholt and Madsen had an issue with 2-torsion. The first correct 2-typical definition for a Witt complex was given by Costeanu.

Definition 5.2. A (big) Witt complex over A is a contravariant functor

$$S \mapsto E_S^\bullet$$

assigning to every subtruncation set of U an anti-symmetric graded ring $E_S^\bullet$ that takes colimits to limits together with a natural ring homomorphism

$$\eta_S : \mathbb{W}_S(A) \to E_S^0$$

and natural maps of graded abelian groups

$$\begin{aligned} d: &\quad E_S^r \to E_S^{r+1} \\ F_n: &\quad E_S^r \to E^r_{\frac{S}{n}} \\ V_n: &\quad E^r_{\frac{S}{n}} \to E_S^r \end{aligned}$$

such that

1. For $x \in E_S^r$, $y \in E_S^t$

$$\begin{aligned} d(x \cdot y) &= d(x) \cdot y + (-1)^r x \cdot d(y) \\ d(d(x)) &= d \log \eta_S([-1]_S) \cdot d(x) \end{aligned}$$

2. For $m, n \in \mathbb{N}$

$$\begin{aligned} F_1 &= V_1 = \mathrm{id} \\ F_m F_n &= F_{nm} \\ V_n V_m &= V_{mn} \\ F_n V_n &= n \cdot \mathrm{id} \\ F_m V_n &= V_n F_m \quad \text{if } (m,n) = 1 \\ F_n \eta_S &= \eta_{\frac{S}{n}} F_n \\ \eta_S V_n &= V_n \eta_{\frac{S}{n}} \end{aligned}$$

3. For all $n \in \mathbb{N}$ the map F_n is a ring homomorphism and F_n and V_n satisfy the projection formula for $x \in E_S^r$ and $y \in E^t_{\frac{S}{n}}$

$$x \cdot V_n(y) = V_n(F_n(x) y).$$

4. For all $n \in \mathbb{N}$ and $y \in E^r_{\frac{S}{n}}$

$$F_n d V_n(y) = d(y) + (n-1) d \log \eta_{\frac{S}{n}}([-1]_{\frac{S}{n}}) \cdot y.$$

5. For all $n \in \mathbb{N}$ and $a \in A$

$$F_n d \eta_S([a]_S) = \eta_{S/n}([a]^{n-1}_{\frac{S}{n}}([a]_{\frac{S}{n}}).$$

A map of Witt complexes is a map of graded rings $f: E_S^\bullet \to \tilde{E}_S^\bullet$ such that

$$\begin{aligned} f \eta_S &= \tilde{\eta} \\ f d &= \tilde{d} f \\ f F_n &= \tilde{F}_n f \\ f V_n &= \tilde{V}_n f. \end{aligned}$$

Part of the structure of a Witt complex is a restriction map

$$R_T^S : E_S^\bullet \to E_T^\bullet$$

for $T \subset S$.

Lemma 5.3. *Every Witt complex is determined, up to canonical isomorphism , on finite truncation sets.*

Proof. For every truncation set S and $r \in \mathbb{N}$ the restriction maps define a bijection

$$E_S^r \to \varprojlim_{T \subset S, \text{ finite}} E_T^r$$

□

In particular, it follows from this that for $a \in \mathbb{W}(A)$ written as a convergent sum $a = \sum_{n\in S} V_n([a_n]_{\frac{S}{n}})$ the element $d\eta_S(a) \in E^1_S$ has a similar representation

$$d\eta_S(a) = \sum_{n\in S} dV_n([a_n]_{\frac{S}{n}}).$$

Remark 5.4. The issue with 2-torsion lies in the appearance of the element $d\log\eta_S([-1]_S)$. This element is annihilated by 2. Indeed, since d is a derivation

$$\begin{aligned}
2d\log\eta_S([-1]_S) &= \frac{d\eta_S([-1]_S)}{\eta_S[-1]_S} + \frac{d\eta_S([-1]_S)}{\eta_S[-1]_S} \\
&= \frac{\eta_S([-1]_S)}{\eta_S([1])} d\eta_S([-1]_S) + \frac{\eta_S([-1]_S)}{\eta_S([1])} d\eta_S([-1]_S) \\
&= \frac{d\eta_S([-1]_S[-1]_S)}{\eta_S([1]_S)} = d\log\eta_S([1]_S) = 0
\end{aligned}$$

It follows that $d\log\eta_S([-1]_S)$ is zero if 2 is invertible or i $2 = 0$ in A because then $[-1]_S = [1]_S$.

Moreover, since

$$[-1]_S = -[1]_S + V_2([1]_{\frac{S}{2}})$$

it follows that $d\log\eta_S([-1]_S)$ is also zero if S contains only odd integers.

We see therefore that in these cases, d is a differential and makes $E^\bullet_S$ into an anitsymmetric differential graded ring.

Lemma 5.5. *Let $m, n \in \mathbb{N}$, and $c = (m, n)$ the greatest common divisor, choose any pair $i, j \in \mathbb{Z}$ such that $mi + nj = c$. The following relations hold for every (big) Witt complex:*

$$\begin{aligned}
dF_n &= nF_n d \\
V_n d &= ndV_n \\
F_m dV_n &= idF_{\frac{m}{c}} V_{\frac{n}{c}} + jF_{\frac{m}{c}} V_{\frac{n}{c}} d + (c-1)d\log\eta_{\frac{S}{m}}([-1]_{\frac{S}{m}}) \cdot F_{\frac{m}{c}} V\frac{n}{c} \\
d\log\eta_S([-1]_S) &= \sum_{r\in\mathbb{N}} 2^{r-1} dV_{2^r}\eta_{\frac{S}{2^r}}([1]_{\frac{S}{2^r}}) \\
d\log\eta_S([-1]_S) \cdot d\log\eta_S([-1]_S) &= 0 \\
dd\log\eta_S([-1]_S) &= 0 \\
F_n(d\log\eta_S([-1]_S)) &= d\log\eta_{\frac{S}{n}}([-1]_{\frac{S}{n}})
\end{aligned}$$

Proof. This follows mostly by explicit calculations. We will do some, and leave the rest as exercise. For the first equation:

$$\begin{aligned}
dF_n(x) &= F_n dV_n F_n(x) - (n-1)d\log\eta[-1] \cdot F_n(x) \quad \text{this follows from (4) of the definition} \\
&= F_n d(V_n\eta([1]) \cdot x) - (n-1)d\log\eta([-1]) \cdot F_n(x) \quad \text{from the projectin formula} \\
&= F_n(dV_n\eta([1]) \cdot x + V_n\eta([1]) \cdot dx) - (n-1)d\log\eta([-1]) \cdot F_n(x) \quad \text{because } d \text{ is a derivation} \\
&= F_n dV_n\eta([1]) \cdot F_n(x) + F_n V_n\eta([1]) \cdot F_n d(x) - (n-1)d\log\eta([-1]) \cdot F_n(x) \\
&= (n-1)d\log\eta([-1]) \cdot F_n(x) + nF_n d(x) - (n-1)d\log\eta([-1]) \cdot F_n(x) \quad \text{from (4) and (2) of the definition} \\
&= n_n d(x)
\end{aligned}$$

The calculation or the second equality is similar and left as an exercise.

Next we proof the last formula.

$$\begin{aligned}
F_n(d\log\eta_S([-1]_S)) &= F_n(\eta_S([-1]_S^{-1}) d\eta_S([-1]_S) \\
&= F_n\eta_S([-1]_S^{-1}) F_n d\eta_S([-1]_S) \\
&= \eta_{\frac{S}{n}}([-1]_{\frac{S}{n}}^{-n}) \eta_{\frac{S}{n}}([-1]^{n-1}) d\eta_{\frac{S}{n}}([-1]_{\frac{S}{n}} \quad \text{from (5) of the definition} \\
&= \eta_{\frac{S}{n}}([-1]_{\frac{S}{n}}^{-1}) d\eta_{\frac{S}{n}}([-1]_{\frac{S}{n}}) = d\log\eta_{\frac{S}{n}}([-1])
\end{aligned}$$

Using the three formulae already proved, we can compute the remaining equalities.

$$\begin{aligned} F_m dV_n(x) &= F_{\frac{m}{c}} F_c dV_c V_{\frac{n}{c}}(x) \\ &= F_{\frac{m}{c}} dV_{\frac{n}{c}}(x) + (c-1) d\log \eta_{\frac{S}{c}}([-1]_{\frac{S}{c}}) \cdot F_{\frac{m}{c}} V_{\frac{n}{c}}(x) \quad \text{with property (4) from the definition} \\ &= ((\frac{m}{c})i + (\frac{n}{c})j) F_{\frac{m}{c}} dV_{\frac{n}{c}}(x) + (c-1) d\log \eta_{\frac{S}{c}}([-1]_{\frac{S}{c}}) \cdot F_{\frac{m}{c}} V_{\frac{n}{c}}(x) \\ &= idF_{\frac{m}{c}} V_{\frac{n}{c}}(x) + jF_{\frac{m}{c}} V_{\frac{n}{c}}(x) + (c-1) d\log \eta_{\frac{S}{c}}([-1]_{\frac{S}{c}}) \cdot F_{\frac{m}{c}} V_{\frac{n}{c}}(x) \end{aligned}$$

The sum formula for $d\log \eta_S([-1]_S)$ follows by induction: We know from an exercise that $[-1]_S = -[1]_s + V_2([1]_{\frac{S}{2}})$. Use this to show that

$$d\log \eta_S([-1]_S) = dV_2 \eta_{\frac{S}{2}}([1]_{\frac{S}{2}}) + V_2(d\log \eta_{\frac{S}{2}}([-1]_{\frac{S}{2}}))$$

then the induction argument is obvious.

Using this, we also find

$$\begin{aligned} dV_2(d\log \eta_{\frac{S}{2}}([-1]_{\frac{S}{2}}) &= \sum_{r\in\mathbb{N}} 2^r ddV_{2^{r+1}} \eta_{\frac{S}{2^{r+1}}}([1]_{\frac{S}{2^{r+1}}}) \\ &= \sum_{r\in\mathbb{N}} 2^r d\log \eta_S([-1]_S) \cdot dV_{2^{r+1}} \eta_{\frac{S}{2^{r+1}}}([1]_{\frac{S}{2^{r+1}}}) \quad \text{because of (1) of the definition} \\ &= 0 \quad \text{because } d\log \eta([-1]) \text{ is annihilated by 2} \end{aligned}$$

With the equality $[-1]_S = -[1]_S + V_2([1]_{\frac{S}{2}})$ one can show (and the reader s encouraged to do this as an exercise)

$$(d\log \eta_S([-1]_S))^2 = dV_2(d\log \eta_{\frac{S}{2}}([-1]_{\frac{S}{2}})) \cdot \eta_S([1]_S - V_2([1]_{\frac{S}{2}})) = 0,$$

which is zero because the first factor is zero by what we just showed.

It follows from this that $(d\eta_S([-1]_S))^2 = 0$ if spell $d\log$ out. As an exercise, use this to show the last equality □

The next proposition wil play an important role in the λ-ring approach to the construction of the big de Rham–Witt complex.

Proposition 5.6. *For every Witt complex $E_S^\bullet$ over A and every $n \in \mathbb{N}$ the diagram*

$$\begin{array}{ccc} \Omega^1_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & E^1_S \\ \downarrow F_n & & \downarrow F_n \\ \Omega^1_{\mathbb{W}_{\frac{S}{n}}(A)} & \xrightarrow{\eta_{\frac{S}{n}}} & E^1_{\frac{S}{n}} \end{array}$$

commutes

Proof. Wlog we can assume that $S = \mathbb{N}$, as the restriction map $R_S^{\mathbb{N}}$ commutes with Frobenius and the map η. Moreover, because a Witt complex is determined on finite truncation sets, and in particular we have a representation for $a \in \mathbb{W}(A)$

$$d\eta_S(a) = \sum_{n\in S} dV_n([a_n]_{\frac{S}{n}})$$

it is enough to show for every $n \in \mathbb{N}$, $p \in \mathbb{N}$ prime and $a \in A$

$$F_p dV_n \eta_{\mathbb{N}}([a]_{\mathbb{N}}) = \eta_{\mathbb{N}} F_p dV_n([a]_{\mathbb{N}})$$

in $E^1_{\mathbb{N}}$.

Case p does not devide n. Set $k = \frac{(1-n^{p-1})}{p}$ and $l = n^{p-2}$. Then $kp + ln = 1$, and $c = (p, n) = 1$ and F_p and V_n commute. Then by the previous lemma

$$\begin{aligned} F_p dV_n \eta([a]) &= k \cdot dV_n F_p \eta([a]) + l \cdot V_n F_p d\eta([a]) \\ &= k \cdot dV_n \eta([a]^p) + l \cdot V_n \eta([a]^{p-1} d[a]) \end{aligned}$$

Now we have to compute $\eta F_p dV_n([a])$. For this we need the equalities

$$F_p db = b^{p-1}db + d\left(\frac{F_p(b) - b^p}{p}\right)$$

and

$$V_m(a)^n = m^{n-1}V_m(a^n)$$

which are left to the reader as exercise.

$$\begin{aligned}
\eta F_p dV_n([a]) &= \eta(V_n([a])^{p-1} \cdot dV_n([a]) + d\left(\frac{F_pV_n[a] - (V_n[a])^p}{p}\right)) \\
&= \eta(n^{p-2} \cdot V_n([a]^{p-1}) \cdot dV_n([a]) + d\left(\frac{V_n([a]^p) - n^{p-1}V_n([a]^p)}{p}\right)) \\
&= \eta(l \cdot V_n([a]^{p-1})dV_n([a]) + kdV_n([a]^p)) \\
&= l \cdot V_n\eta([a]^{p-1})dV_n\eta([a]) + k \cdot dV_n\eta([a]^p) \\
&= l \cdot V_n\left(\eta([a]^{p-1}) \cdot F_n dV_n\eta([a])\right) + k \cdot dV_n\eta([a]^p) \quad \text{because of the projection formula} \\
&= l \cdot V_n\eta([a]^{p-1}d[a]) + k \cdot dV_n\eta([a]^p) \quad \text{because of (4) if the definition and } n^{p-2}(n-1)d\log\eta([-1]) = 0
\end{aligned}$$

Case p divides n. In this case, one treats $p = 2$ and p odd separately. This will b done in the exercise session. □

In order to extend this diagram – and in particular the morphism η to complexes, we have to modify the usual complex Ω.

Remark 5.7. Note that the Frobenius $F_n : \Omega^1_{\mathbb{W}_S(A)} \to \Omega^1_{\mathbb{W}_{\frac{S}{n}}(A)}$ is not the one following from functoriality, but it is off by a constant factor. We will discuss the existence of such a Frobenius later on.

5.2 Two anticommutative graded algebras

The big de Rham–Witt complex is closely related to K-theory. In fact, it was introduces by Hesselholt and Madsen in order to give an algebraic description of the equivariant homotopy groups in low degrees of Bökstedt's topological Hochschild spectrum of a commutative ring. This functorial algebraic description is essential to understand algebraic K-theory by means of the cyclotomic trace map of Bökstedt–Hsiang–Madsen. Recall that for a field an easy description of Quillen K-theory up to degree 2 is given by Milnor K-theory. Therefore, we should not necesserily expect the big de Rham–Witt complex to be made up of alterating forms, but rather some sort of Steinberg relation should be saitsfied. This leads to the following definition.

Definition 5.8. Let A be a ring. The graded $\mathbb{W}(A)$-algebra

$$\widehat{\Omega}_{\mathbb{W}(A)} := T_{\mathbb{W}(A)}\Omega^1_{\mathbb{W}(A)}/J$$

is the quotient of the tensor algebra of the $\mathbb{W}(A)$-module $\Omega^1_{\mathbb{W}(A)}$ by the graded ideal generated by the elements of the form

$$da \otimes da - d\log[-1] \otimes F_2(da)$$

for $a \in \mathbb{W}(A)$.

The defining relation $da \cdot da = d\log[-1] \cdot F_2(da)$ is analogous to the Steinberg relation in Milnor K-theory. (For $a \in A$ this corresponds to

$$d\log[a] \cdot d\log[a] = d\log[-1]d\log[a]$$

which we compare to the relation $\{a, a\} = \{-1, a\}$ in Milnor K-theory.)

We will mention some of the important properties of this construct (and show some of them).

Lemma 5.9. *The graded $\mathbb{W}(A)$-algebra $\widehat{\Omega}_{\mathbb{W}(A)}$ is anticommutative.*

Proof. We have to show that for $a, b \in \mathbb{W}(A)$ the sum $da \cdot db + db \cdot da \in \widehat{\Omega}^2_{\mathbb{W}(A)}$ equals zero. we compute first using the defining relations in two ways:

$$d(a+b) \cdot d(a+b) = d\log[-1] \cdot F_2 d(a+b) = d\log[-1] \cdot F_2 da + d\log[-1] \cdot F_2 db$$

and

$$d(a+b) \cdot d(a+b) = da \cdot da + da \cdot db + db \cdot da + db \cdot db = d\log[-1] \cdot F_2 da + da \cdot db + db \cdot da + d\log[-1] \cdot F_2 db$$

Comparing the two expressions shows that $db \cdot da = da \cdot db$. □

Proposition 5.10. *There exists a unique graded derivation*

$$d : \widehat{\Omega}_{\mathbb{W}(A)} \to \widehat{\Omega}_{\mathbb{W}(A)}$$

extending the derivation $d : \mathbb{W}(A) \to \Omega^1_{\mathbb{W}(A)}$ *and satisfying*

$$dd\omega = d\log[-1] \cdot d\omega.$$

Moreover, the element $d\log[-1]$ *is a cycle.*

Proof. Inductively, the map d will be given for $a_0, \dots, a_q \in \mathbb{W}(A)$

$$d(a_0 da_1 \cdots da_q) = da_0 \cdots da_q + qd\log[-1] \cdot a_0 da_1 \cdots da_q$$

whoch means that the second summand disappears for q even and equals $d\log[-1] \cdot a_0 da_1 \cdots da_q$ for q odd. If the so defined map is a well defined graded derivation satisfying the relation $dd\omega = d\log[-1] \cdot d\omega$, it is necessarily unique. This is left to the reader as exercise.

It then follows from $dd\omega = d\log[-1] \cdot d\omega$ that $d\log[-1]$ is in fact a cycle:

$$\begin{aligned} d(d\log[-1]) &= d([-1]d[-1]) \\ &= d[-1] \cdot d[-1] + [-1]dd[-1] \\ &= d\log[-1] \cdot F_1 d[-1] + [-1]d\log[-1]d[-1] \\ &= d\log[-1] \cdot [-1]d\log[-1] + [-1]d\log[-1]d[-1] \\ &= 2(d\log[-1] \cdot [-1]d[-1]) = 0 \end{aligned}$$

(because $\widehat{\Omega}_{\mathbb{W}(A)}$ is anticommutative). □

Note that in general there is no $\mathbb{W}(A)$-algebra map $\widehat{\Omega}_{\mathbb{W}(A)} \to \Omega_{\mathbb{W}(A)}$ compatible with the derivations!

Proposition 5.11. *Let A be a ring and $n \in \mathbb{N}$. There is a unique homomorphism of graded rings*

$$F_n : \widehat{\Omega}_{\mathbb{W}(A)} \to \widehat{\Omega}_{\mathbb{W}(A)}$$

extending F_n from degree 0 *and* 1*. Additionally*

$$dF_n = nF_n d.$$

Proof. Similar to th definition of d, the map F_n has to be given by

$$F_n(a_0 da_1 \cdots da_q) = F_n(a_0) F_n(da_1) \dots F_n(da_q)$$

to be a graded ring homomorphism extending F_n from degrees 0 and 1, and this is unique if it is well defined. To show this, one has to sow that

$$F_n(da) F_n(da) = F_n(d\log[-1]) F_n(F_2 da)$$

It suffices to show this for $n = p$ prime. This is left to the reader.

The formula $dF_n = nF_n d$ is already known in degree 1. Again, wlog, we can assume $n = p$ to be prime. To extend this to higher degrees, let $a \in \mathbb{W}(A)$. Then

$$dF_p(da) = d(a^{p-1}da + d\left(\frac{F_p(a) - a^p}{p}\right) = (p-1)a^{p-2}dada + d\log[-1] \cdot F_p da$$

which is 0 for $p = 2$ by the defining relations, and equal to $d\log[-1] \cdot F_p da$ of p is odd (because then $p - 1$ is even which kills the first summand). Induction give the formula for higher degrees than 2. □

So far, we hae established some important additional structures on $\widehat{\Omega}_{\mathbb{W}(A)}$ however, Verschiebung does in general not extend to this $\mathbb{W}(A)$ algebra. We therefore define a quotient of it, where in degree 1 the desired relation between Verschiebung, Frobenius and derivation holds.

Definition 5.12. Let A be a ring. Set

$$\check{\Omega}_{\mathbb{W}(A)} = \widehat{\Omega}_{\mathbb{W}(A)}/K$$

where K is the graded ideal generated by the elements

$$F_p dV_p(a) - da - (p-1)d\log[-1] \cdot a$$

for all primes p and all $a \in \mathbb{W}(A)$. This is a graded $\mathbb{W}(A)$-algebra.

Note that the element $F_p dV_p(a) - da - (p-1)d\log[-1] \cdot a$ is annihilated by p (in particular, it is zero if p is invertible in A and hence in $\mathbb{W}(A)$).

In order for this definition to be useful, the maps F_n and d should descent from $\widehat{\Omega}_{\mathbb{W}(A)}$.

Lemma 5.13. *For all $n \in \mathbb{N}$ the Frobenius map $F_n : \widehat{\Omega}_{\mathbb{W}(A)} \to \widehat{\Omega}_{\mathbb{W}(A)}$ induces a map of graded algebras*

$$F_n : \check{\Omega}_{\mathbb{W}(A)} \to \check{\Omega}_{\mathbb{W}(A)}.$$

The graded derivation $d : \widehat{\Omega}_{\mathbb{W}(A)} \to \widehat{\Omega}_{\mathbb{W}(A)}$ induces a graded derivation

$$d : \check{\Omega}_{\mathbb{W}(A)} \to \check{\Omega}_{\mathbb{W}(A)}.$$

Moreover, or all $n \in \mathbb{N}$ and $a \in \mathbb{W}(A)$

$$F_n dV_n(a) = da + (n-1)d\log[-1] \cdot a$$

holds in $\check{\Omega}^1_{\mathbb{W}(A)}$.

Proof. The calculations to do here are not difficult, and in general obvious, but a bit tedious. □

So far, the definitions hold for the big Witt vectors, meaning that $S = \mathbb{N}$. But using restriction, the other cases are covered as well.

Definition 5.14. Let A be a ring, $S \subset \mathbb{N}$ a truncation set and $I_S(A) \subset \mathbb{W}(A)$ the kernel of $R^{\mathbb{N}}_S : \mathbb{W}(A) \to \mathbb{W}_S(A)$. The maps

$$\widehat{\Omega}_{\mathbb{W}(A)} \xrightarrow{R^{\mathbb{N}}_S} \widehat{\Omega}_{\mathbb{W}_S(A)} \quad \text{and} \quad \check{\Omega}_{\mathbb{W}(A)} \xrightarrow{R^{\mathbb{N}}_S} \check{\Omega}_{\mathbb{W}_S(A)}$$

are the quotient maps that annihilate the respective graded ideals generated by $I_S(A)$ and $dI_S(A)$.

Lemma 5.15. *The derivation, restriction and Frobenius defined before induce maps*

$$\begin{array}{rlrl} d: & \widehat{\Omega}_{\mathbb{W}_S(A)} \to \widehat{\Omega}_{\mathbb{W}_S(A)} & d: & \check{\Omega}_{\mathbb{W}_S(A)} \to \check{\Omega}_{\mathbb{W}_S(A)} \\ R^T_S: & \widehat{\Omega}_{\mathbb{W}_S(A)} \to \widehat{\Omega}_{\mathbb{W}_T(A)} & R^T_S: & \check{\Omega}_{\mathbb{W}_S(A)} \to \check{\Omega}_{\mathbb{W}_T(A)} \\ F_n: & \widehat{\Omega}_{\mathbb{W}_S(A)} \to \widehat{\Omega}_{\mathbb{W}_{\frac{S}{n}}(A)} & F_n: & \check{\Omega}_{\mathbb{W}_S(A)} \to \check{\Omega}_{\mathbb{W}_{\frac{S}{n}}(A)} \end{array}$$

The maps d are graded derivations, the maps R^T_S and F_n are graded ring homomorphisms; R^T_S and d commute and $dF_n = nF_n d$.

Proof. For the first part, there are a few equations to check. The second part is clear. □

Now we want to extend the commuting diagram for a Witt complex E_S

$$\begin{array}{ccc} \Omega^1_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & E^1_S \\ \downarrow F_n & & \downarrow F_n \\ \Omega^1_{\mathbb{W}_{\frac{S}{n}}(A)} & \xrightarrow{\eta_{\frac{S}{n}}} & E^1_{\frac{S}{n}} \end{array}$$

to $\check{\Omega}_{\mathbb{W}_S(A)}$.

Proposition 5.16. *Let E_S be a Witt complex over the ring A. There is a unique natural homomorphism of graded rings*

$$\eta_S : \check{\Omega}_{\mathbb{W}_S(A)} \to E_S$$

that extends the natural ring homomorphism $\eta_S : \mathbb{W}_S(A) \to E_S^0$ and commutes with derivations. For $m \in \mathbb{N}$ the diagram

$$\begin{array}{ccc} \check{\Omega}_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & E_S \\ \downarrow{\scriptstyle F_m} & & \downarrow{\scriptstyle F_m} \\ \check{\Omega}_{\mathbb{W}_{\frac{S}{m}}(A)} & \xrightarrow{\eta_{\frac{S}{m}}} & E_{\frac{S}{m}} \end{array}$$

commutes.

Proof. As before, there is no other way the map η_S can be given than by

$$\eta_S(a_0 da_1 \cdots da_q) = \eta_S(a_0) d\eta_S(a_1) \cdots d\eta_S(a_q)$$

To show that it is well defined, we note first from the proposition in degree 1 that

$$F_2 d\eta_{\mathbb{N}}(a) = \eta_{\mathbb{N}} F_2 d(a) = \eta_{\mathbb{N}} \left(a da + d \left(\frac{F_2(a) - a^2}{2} \right) \right) = \eta_{\mathbb{N}}(a) d\eta_{\mathbb{N}}(a) + d\eta_{\mathbb{N}} \left(\frac{F_2(a) - a^2}{2} \right)$$

Now we apply d to this equation, so that the left hand side becomes

$$dF_2 d\eta_{\mathbb{N}}(a) = 2F_2 dd\eta_{\mathbb{N}}(a) = 0$$

and the right hand side reads

$$d\eta_{\mathbb{N}}(a) d\eta_{\mathbb{N}}(a) + d\log \eta_{\mathbb{N}}([-1]_{\mathbb{N}}) \cdot (\eta_{\mathbb{N}}(a) d\eta_{\mathbb{N}}(a) + d\eta_{\mathbb{N}} \left(\frac{F_2(a) - a^2}{2} \right)) = d\eta_{\mathbb{N}}(a) d\eta_{\mathbb{N}}(a) d\log \eta_{\mathbb{N}}([-1]_{\mathbb{N}}) \cdot F_2 d\eta_{\mathbb{N}}(a)$$

and together the equation

$$0 = d\eta_{\mathbb{N}}(a) d\eta_{\mathbb{N}}(a) d\log \eta_{\mathbb{N}}([-1]_{\mathbb{N}}) \cdot F_2 d\eta_{\mathbb{N}}(a)$$

which is the defining relation of $\widehat{\Omega}_{\mathbb{W}_S(A)}$. Thus the above defined map is well defined on $\widehat{\Omega}_{\mathbb{W}_S(A)} \to E_S$. Moreover this map factors through $\check{\Omega}_{\mathbb{W}_S(A)}$ which is the quotient of $\widehat{\Omega}_{\mathbb{W}_S(A)}$ by the ideal generated by $F_p dV_p(a) - da - (p-1) d\log[-1] \cdot a$ because o point (4) of the definition of Witt complexes. Finally it is clear from the definition of η_S above, and from the equivalent result in degree 1, that the desired diagram commutes. ☐

The existence of the Forbenius used here follows quite explicitely from the theory λ-rings, and modules and derivations over those, which will be the subject of the following section.

5.3 Modules and derivations over λ-rings

We already mentioned the following fact, when we introduced the big Witt vectors. For simplicity, denote $\mathbb{W}(A) := \mathbb{W}_{\mathbb{N}}(A)$ for a ring A as above.

Proposition 5.17. *There exists a unique natural ring homomorphism*

$$\Delta = \Delta_A : \mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(A))$$

such that for any $n \in \mathbb{N}$

$$w_n \circ \Delta = F_n : \mathbb{W}(A) \to \mathbb{W}(A).$$

In addition, the following diagrams, with $\varepsilon_B = w_1 : \mathbb{W}(B) \to B$ for a ring B, commute

$$\begin{array}{ccccc} \mathbb{W}(A) & \xleftarrow{\varepsilon_{\mathbb{W}(A)}} & \mathbb{W}(\mathbb{W}(A)) & \xrightarrow{\mathbb{W}(\varepsilon_A)} & \mathbb{W}(A) \\ & \Vert & \uparrow{\scriptstyle \Delta_A} & \Vert & \\ & & \mathbb{W}(A) & & \end{array}$$

and

$$\begin{array}{ccc} \mathbb{W}(\mathbb{W}(\mathbb{W}(A))) & \xleftarrow{\Delta_{\mathbb{W}(A)}} & \mathbb{W}(\mathbb{W}(A)) \\ \uparrow{\scriptstyle \mathbb{W}(\Delta_A)} & & \uparrow{\scriptstyle \Delta_A} \\ \mathbb{W}(\mathbb{W}(A)) & \xleftarrow{\Delta_A} & \mathbb{W}(A) \end{array}$$

Proof. To prove existence, it is enough to do that in the universal case $A = \mathbb{Z}[a_1, a_2, \dots]$ and $a = (a_1, a_2, \dots)$ there is an element $\Delta_A(a) \in \mathbb{W}(\mathbb{W}(A))$ with image under the ghost map

$$w : \mathbb{W}(\mathbb{W}(A)) \to \mathbb{W}(A)^{\mathbb{N}}$$

is $(F_n(a))_{n\in\mathbb{N}}$. Since w in this universal case is injective, the element $\Delta_A(a)$ is unique - if it exists.

By Dworks Lemma and the definition of F_p, $(F_n(a))$ is in the image of the ghost map, iff for $p \in \mathbb{N}$ prime and $n \in p\,\mathbb{N}$

$$F_n(a) \equiv F_p(F_{\frac{n}{p}}) \bmod p^{\nu_p(n)}\, \mathbb{W}(A),$$

which follows from $F_n([a]_S) = [a]^n_{\frac{S}{n}}$.

Thus existence and uniqueness of the map Δ. One checks that the diagrams commute by computing them in ghost coordinates. □

Note that the map $\Delta_n : \mathbb{W}(A) \to \mathbb{W}(A)$ given by the n$^{\text{th}}$ component of Δ is in general not a ring homomorphism.

Moreover, for $a \in A$: $\Delta([a]) = [[a]]$.

This natural transformation is called the universal λ-operation. With this, Grothendieck's definition of λ-rings can be stated as follows.

Definition 5.18. A λ-ring is a pair (A, λ), where A is a ring, and $\lambda : A \to \mathbb{W}(A)$ such that the diagrams

$$\begin{array}{ccc} A & \xleftarrow{\varepsilon_A} & \mathbb{W}(A) \\ & \searrow\!\!\!\!\!\!= & \uparrow{\scriptstyle \lambda} \\ & & A \end{array}$$

and

$$\begin{array}{ccc} \mathbb{W}(\mathbb{W}(A)) & \xleftarrow{\Delta_A} & \mathbb{W}(A) \\ \uparrow{\scriptstyle \mathbb{W}(\lambda)} & & \uparrow{\scriptstyle \lambda} \\ \mathbb{W}(A) & \xleftarrow{\lambda} & A \end{array}$$

commute. A morphism of λ-rings $f : (A, \lambda_A) \to (B, \lambda_B)$ is a ring homomorphism $f : A \to B$ such that

$$\lambda_B \circ f = \mathbb{W}(f) \circ \lambda_A.$$

For a λ-ring (A, λ) we denote by $\lambda_n : A \to A$ the n^{th} Witt component of $\lambda(a)$. The so defined map is in general neither additive nor multiplicative.

Definition 5.19. Let (A, λ) be a λ-ring. The associated n^{th} Adams operation is the composite ring homomorphisms

$$\psi_n = w_n \circ \lambda : A \to A.$$

We mention some results:

Lemma 5.20. *Let (A, λ) be a λ-ring. The associated Adams operations satisfy:*

1. *the map* $\psi_1 = \mathrm{id}_A$
2. *for all positive integers* $m, n \in \mathbb{N}$*:* $\psi_m \circ \psi_n = \psi_{mn}$
3. *for a prime* $p \in \mathbb{N}$*,* $a \in A$*:* $\psi_p(a) \equiv a^p \bmod pA$

Proof. The properties (1) and (3) follow directly from the definition. (2) follows from

$$\begin{aligned}\psi_m \circ \psi_n &= w_m \circ \lambda \circ w_n \circ \lambda \\ &= w_m \circ w_n \circ \mathbb{W}(\lambda) \circ \lambda \quad \text{from naturality of } w_n \\ &= w_m \circ w_n \circ \Delta \circ \lambda \quad \text{by definition of a } \lambda\text{-ring} \\ &= W_m \circ F_n \circ \lambda \quad \text{by definition of } \Delta \\ &= w_{mn} \circ \lambda = \psi_{mn} \quad \text{by definition of } F_n\end{aligned}$$

□

Proposition 5.21 (Wilkerson)**.** *If A is a flat ring over $\mathbb{Z}$, with a family of ring endomorphisms ψ_n satisfying properties (1)-(3) from the previous lemma. Then there is a unique λ-ring structure on A for which the ψ_n are the associated Adams operations.*

Proof. This can be found in [8]. □

Lastly, we cite a result obtained independently by Borger [2, 3] and van der Kallen [7].

Theorem 5.22. *Let $f : A \to B$ be étale, S a finite truncation set, $n \in \mathbb{N}$. Then the induced morphism*

$$\mathbb{W}_S(f) : \mathbb{W}_S(A) \to \mathbb{W}_S(B)$$

is étale and the diagram

$$\begin{array}{ccc} \mathbb{W}_S(A) & \xrightarrow{\mathbb{W}_S(f)} & \mathbb{W}_S(B) \\ \downarrow{\scriptstyle F_n} & & \downarrow{\scriptstyle F_n} \\ \mathbb{W}_{\frac{S}{n}}(A) & \xrightarrow{\mathbb{W}_{\frac{S}{n}}(f)} & \mathbb{W}_{\frac{S}{n}}(B) \end{array}$$

is cocartesian.

The definition of modules over λ-rings used by Hesselholt in [4, Sec. 2] is based on the following definition employed by Beck [1] in his thesis.

Let $\mathcal{C}$ be a category with finite limits and $X \in \mathcal{C}$. Then the category of X-modules $(\mathcal{C}/X)_{\mathrm{ab}}$ is the category of abelian group objects in $\mathcal{C}$ over X. The derivations from X to the X-module $(Y/X, +_Y, 0_Y, -_Y)$ is the set

$$\mathrm{Der}(X, (Y/X, +_Y, 0_Y, -_Y)) = \mathrm{Hom}_{\mathcal{C}/X}(X/X, Y/X).$$

We will use this as a working definition.

Remark 5.23. A few reminders about category theory.

In general an adjunction from a category $\mathscr{C}$ to a category $\mathscr{D}$ is a quadruple $(F, G, \varepsilon, \eta)$ where $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are functors, and $\varepsilon : F \circ G \Rightarrow \mathrm{id}$ and $\eta : G \circ F \Rightarrow \mathrm{id}$ are natural transformations, such that

$$F \xRightarrow{F\circ\eta} F \circ G \circ F \xRightarrow{\varepsilon\circ F} F \qquad \text{and} \qquad G \xRightarrow{\eta\circ G} G \circ F \circ G \xRightarrow{G\circ\varepsilon} G$$

are equal to the respective identity natural transformation. This is often refer to as triangle identities. The transformations ε and η are called counit and unit of the adjunction. The adjunction is calle adjoint equivalence, if they are both isomorphisms.

A functor $G : \mathscr{D} \to \mathscr{C}$ admits a left adjoint if an adjunction $(F, G, \varepsilon, \eta)$ exists. F is then called a left adjoint of R. If a left adjoint exists, then it is unique up to unique isomorphism. Similar for right adjoints.

Let $\mathscr{A}$ be the category of (commutative) rings. For $A \in \mathscr{A}$ we define an adjunction $(F, G, \varepsilon, \eta)$ from the category $(\mathscr{A}/A)_{\mathrm{ab}}$ of A-modules as defined above (abelian group objects in the category $\mathscr{A}/A$), to the category $\mathscr{M}(A)$ of A-modules in the usual sense:

Let $f : B \to A$ be in $\mathscr{A}/A$ and the abelian group structure given by

$$\begin{array}{ccc} B \times_f B & \xrightarrow{+_B} & B \\ \downarrow & & \downarrow{\scriptstyle f} \\ A & = & A \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{0_B} & B \\ \| & & \downarrow{\scriptstyle f} \\ A & = & A \end{array} \qquad \begin{array}{ccc} B & \xrightarrow{-_B} & B \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ A & = & A \end{array}$$

Then F associates to the abelian group object $(f, +_B, 0_B, -_B)$ the A-module $M = \mathrm{Ker}(f)$ with the A-module structure

$$a.x = 0_B(a)x.$$

On the other hand, if M is an A-module, let $A \ltimes M$ be the ring given by $A \oplus M$ with multiplication

$$(a, x).(a', x') = (aa', ax' + a'x)$$

and let $G(M)$ be the group object $(f, +, 0, -)$ with $f : A \ltimes M \to A$ the projection, $(a, x) + (a, x') = (a, x + x')$, $0(a) = (a, 0)$ and $-(a, x) = (a, -x)$. We define $\varepsilon : G \circ F \Rightarrow \mathrm{id}$ and $\eta : F \circ G \Rightarrow \mathrm{id}$ by

$$\varepsilon(a, x) = 0_B(a) + x \quad \text{and} \quad \eta(x) = (0, x)$$

Lemma 5.24. *If A is a ring, then the quadruple $(F, G, \eta, \varepsilon)$ is an adjoint equivalence of categories from $(\mathscr{A}/A)_{ab}$ to $\mathscr{M}(A)$.*

Proof. This is a result due to Beck and will be done in the exercise session. □

We will look at the analogous statement for λ-rings.

Before, we will study the Witt vectors of the ring $A \ltimes M$ defined earlier. Recall that the polynomials $s_n(\underline{a}, \underline{b}), p_n(\underline{a}, \underline{b}), i_n(\underline{a})$ which define the sum product and inverse in the ring of (big) Witt vectors have constant term 0. Thus the (big) Witt vectors can be defined for non-unital rings as well. Moreover, by induction one sees that they are congruent to

$$\begin{aligned} s_n(\underline{a}, \underline{b}) &\equiv a_n + b_n \\ p_n(\underline{a}, \underline{b}) &\equiv a_n b_n \\ i_n(\underline{a}) &\equiv -a_n \end{aligned}$$

modulo higher degrees. If we consider the module M as non-unital ring with zero multiplication, then its Witt ring $\mathbb{W}_S(M)$ has also zero multiplication, and has underlying additive group M^S with component-wise addition.

Similarly, one shows, that the polynomials defining the Frobenius and the universal λ-operation have constant term zero and are congruent to na_{nm} for F_n and a_{nm} for Δ_n resepectively, so that

$$\begin{aligned} F_n : &\quad \mathbb{W}_S(M) \to \mathbb{W}_{\frac{S}{n}}(M), \ (x_m)_{m \in S} \mapsto (nx_{nm})_{m \in \frac{S}{n}} \\ \Delta_M : &\quad \mathbb{W}(M) \to \mathbb{W}(\mathbb{W}(M)), \ (x_m)_{m \in \mathbb{N}} \mapsto ((x_{me})_{e \in \mathbb{N}})_{m \in \mathbb{N}} \end{aligned}$$

Lemma 5.25. *Let S be a truncation set, A a ring and M an A-module. Assume that $\mathbb{W}_S(M)$ is endowed witht the $\mathbb{W}_S(A)$-module structure such that for $a \in \mathbb{W}_S(A)$ and $x \in \mathbb{W}_S(M)$, $ax \in \mathbb{W}_S(M)$ has Witt components $(ax)_n = w_n(a)x_n$. Then the canonical inclusions $i_1 : A \to A \ltimes M$ and $i_2 : M \to A \ltimes M$ induce a ring isomorphism*

$$i_{1*} + i_{2*} : \mathbb{W}_S(A) \ltimes \mathbb{W}_S(M) \to \mathbb{W}_S(A \ltimes M).$$

Proof. Consider the diagram of rings

$$0 \longrightarrow M \xrightarrow{i_2} A \ltimes M \underset{i_1}{\overset{p_1}{\rightleftarrows}} A \longrightarrow 0$$

Although not a priori exact as diagram of rings, it is split exact seen as diagram of additive groups. Likewise, the induced diagram of rings

$$0 \longrightarrow \mathbb{W}_S(M) \xrightarrow{i_{2*}} \mathbb{W}_S(A \ltimes M) \underset{i_{1*}}{\overset{p_{1*}}{\rightleftarrows}} \mathbb{W}_S(A) \longrightarrow 0$$

has an underlying diagram of additive groups which is split exact. It follows that the map of the statement is an isomorphism of additive groups. Moreover, it is a morphism of rings, if $\mathbb{W}_S(M)$ is given the $\mathbb{W}_S(A)$-module structure such that $i_{2*}(ax) = i_{1*}(a)i_{2*}(x)$ for all $a \in \mathbb{W}_S(A)$ and $x \in \mathbb{W}_S(M)$. It remains to show that ax equals the Witt vector y with components $w_n(a)x_n$. Wlog, we may assume that A and M are torsion free (otherwise, we can find a surjection from a torsion free ring). In this case, the ghost map is injective, so that we can use ghost components to show the claim. In other words, for each $n \in \mathbb{N}$

we have to show $w_n(ax) = w_n(y)$ in $\mathbb{W}_S(M)$, which means we have to show $i_2(w_n(ax)) = i_2(w_n(y))$ in $\mathbb{W}_S(A \ltimes M)$. Bearing in mind that w_n is a ring homomorphism we compute

$$\begin{aligned} i_2(w_n(ax)) &= w_n(i_{2*}(ax)) \\ &= w_n(i_{1*}(a)i_{2*}(x)) \\ &= w_n(i_{1*}(a))w_n(i_{2*}(x)) \\ &= i_1(w_n(a))i_2(w_n(x)) \\ &= i_2(w_n(a)w_n(x)) \\ &= i_2(nw_n(a)x_n) \\ &= i_2(ny_n) = i_2(w_n(y)) \end{aligned}$$

which proves the claim. □

To describe the elements of $\mathbb{W}_S(A \ltimes M)$ we prove the following:

Lemma 5.26. *Let A, M, S be as above, $a \in \mathbb{W}_S(A)$ and $x \in \mathbb{W}_S(M)$. Then the Witt components $b_n = a_n.y_n \in A \ltimes M$ of $b = i_{1*}(a) + i_{2*}(x) \in \mathbb{W}_S(A \ltimes M)$ satisfy*

$$\sum_{e|n} a_e^{\frac{n}{e}-1} y_e = x_n.$$

Proof. This is an exercise. □

Inspired by this, we now consider for a ring A and an A-module M and truncation set S the $\mathbb{W}_S(A)$-module $\mathbb{W}_S(M)$ to be the set M^S with component wise addition and with scalar multiplication defined for $a \in \mathbb{W}_S(A), x \in \mathbb{W}_S(M)$ by

$$(ax)_n = \psi_{A,n}(a)x_n$$

where $\psi_{A,n}$ is the n^{th} Adams operation of A.

Remark 5.27. In the case, when M is the A-module A itself, then the $\mathbb{W}_S(A)$-modules $\mathbb{W}_S(M)$ defined as above is in general not the same as the $\mathbb{W}_S(A)$-module $\mathbb{W}_S(A)$ via multiplication.

Now back to our goal to prove a λ-ring equivalent of Lemma 5.24. For this, we first give a straight forward definition of modules in this context.

Definition 5.28. Let (A, λ_A) be a λ-ring. An (A, λ_A)-module is a pair $(M, \lambda_M)$where M is an A-module and

$$\lambda_M :\to \mathbb{W}(M)$$

a λ_A-linear map such that the diagrams

$$\begin{array}{ccc} M & \xleftarrow{\varepsilon_M} & \mathbb{W}(M) \\ & \diagdown\!\!\diagdown & \uparrow{\scriptstyle \lambda_M} \\ & & M \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathbb{W}(\mathbb{W}(M)) & \xleftarrow{\Delta_M} & \mathbb{W}(M) \\ \uparrow{\scriptstyle \mathbb{W}(\lambda_M)} & & \uparrow{\scriptstyle \lambda_M} \\ \mathbb{W}(M) & \xleftarrow{\lambda_M} & M \end{array}$$

commute.

A morphism $h : (M, \lambda_M) \to (N, \lambda_N)$ of (A, λ_A)-modules is an A-linear map $h : M \to N$ such that

$$\lambda_N \circ h = \mathbb{W}(h) \circ \lambda_M.$$

Denote by $\mathscr{M}(A, \lambda_A)$ the category of (A, λ_A)-modules.

Example 5.29. For a λ-ring (A, λ_A) one can define an (A, λ_A)-module by setting $(M, \lambda_M) = (A, \psi_A)$. Note however, that (A, λ_A) itself is in general not an (A, λ_A)-module.

As we have seen for a ring A $(\mathbb{W}(A), \Delta_A)$ is a λ-ring. In fact, the functor, $R : A \mapsto (\mathbb{W}(A), \Delta_A)$ is right adjoint to the forgetful functor

$$U : \mathscr{A}_\lambda \to \mathscr{A}$$

(with unit given by $\lambda : (A, \lambda_A) \to (\mathbb{W}(A), \Delta_A)$ and counit by $\varepsilon_A : \mathbb{W}(A) \to A$).

We also have an adjunction

$$\mathscr{A}_\lambda/(A, \lambda_A) \underset{R_{(A,\lambda_A)}}{\overset{U_{(A,\lambda_A)}}{\rightleftarrows}} \mathscr{A}/A$$

where the forgetfulfunctor $U_{(A,\lambda_A)}$ takes $f : (B, \lambda_B) \to (A, \lambda_A)$ to $f : B \to A$ and its right adjoint takes $f : B \to A$ to the pullback $p_2 : (C, \lambda_C) \to (A, \lambda_A)$ with

$$\begin{array}{ccc} (C, \lambda_C) & \xrightarrow{p_1} & (\mathbb{W}(B), \Delta_B) \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \mathbb{W}(f)} \\ (A, \lambda_A) & \xrightarrow[\lambda_A]{} & (\mathbb{W}(A), \Delta_A) \end{array}$$

Since both functors preserve limits, as the functors above, they induce an adjunction on the subcategory of abelian group objects

$$(\mathscr{A}_\lambda/(A, \lambda_A))_{\mathrm{ab}} \rightleftarrows (\mathscr{A}/A)_{\mathrm{ab}}$$

which correspond to the adjunction

$$\mathscr{M}(A, \lambda_A) \underset{R'}{\overset{U'}{\rightleftarrows}} \mathscr{M}(A)$$

$$(M, \lambda_M) \longmapsto M$$

$$(\lambda_{A*}(\mathbb{W}(N)), \Delta_N) \longleftarrow\!\shortmid N$$

The notation $\lambda_{A*}(\mathbb{W}(N))$ means the $\mathbb{W}(A)$-modules $\mathbb{W}(N)$ considered as an A module via λ_A.

We now come to the analogue of Beck's result.

Proposition 5.30. *Let (A, λ_A) be a λ-ring. There exist a unique adjunction (up to unique isomorphism)*

$$(F^\lambda, G^\lambda, \varepsilon^\lambda, \eta^\lambda) : (\mathscr{A}_\lambda/(A, \lambda_A))_{ab} \to \mathscr{M}(A, \lambda_A)$$

such that in the diagram below the square of left adjoint functors commutes

$$\begin{array}{ccc} (\mathscr{A}/A)_{ab} & \underset{G}{\overset{F}{\rightleftarrows}} & \mathscr{M}(A) \\ {\scriptstyle U_{(A,\lambda_A)}}\uparrow\downarrow{\scriptstyle R_{(A,\lambda_A)}} & & {\scriptstyle U'}\uparrow\downarrow{\scriptstyle R'} \\ (\mathscr{A}_\lambda/(A, \lambda_A))_{ab} & \underset{G^\lambda}{\overset{F^\lambda}{\rightleftarrows}} & \mathscr{M}(A, \lambda_A) \end{array}$$

Moreover, this defines an equivalence of categories.

Proof. Recall that F was defined by associating to an abelian group object $(f : B \to A, +_B, 0_b, -_B)$ the A-module $M = \ker f$ with the module structure $a\dot{y} = 0_B(a)x$. And G was defined by sending an A-module M to the group object $(f : A \ltimes M \to A, +, 0, -)$.

Now let $(f : (B, \lambda_B) \to (A, \lambda_A), +_B, 0_B, -_B) \in (\mathscr{A}_\lambda/(A, \lambda_A))_{\mathrm{ab}}$, then $F^\lambda(f, +_B, 0_B, -_B) = (M, \lambda_M)$ with $M = F(f)$ and $\lambda_M : M \to \mathbb{W}(M)$ induced by functoriality on the kernels of the vertical maps in

$$\begin{array}{ccc} B & \xrightarrow{\lambda_B} & \mathbb{W}(B) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathbb{W}(f)} \\ A & \xrightarrow{\lambda_A} & \mathbb{W}(A) \end{array}$$

and it is clear that $U' \circ F^\lambda = F \circ U_{(A,\lambda_A)}$.

Conversely, for an (A, λ_A)-module (M, λ_M), let $G^\lambda(M, \lambda_M)$ be $G(M)$ of above (with underlying ring $B = A \ltimes M$), with the *lambda*-ring structure $\lambda_B : B \to \mathbb{W}(B)$ given by

$$A \ltimes M \xrightarrow{\lambda_A \oplus \lambda_M} \mathbb{W}(A) \ltimes \mathbb{W}(M) \xrightarrow{i_1 + i_2} \mathbb{W}(A \ltimes M)$$

One then has to show that G^λ is well-defined, for which one needs the three following steps:

1. (B, λ_B) is a λ-ring.
2. The canonical projection $f : (B, \lambda_B) \to (A, \lambda_A)$ is a λ-ring morphism.
3. The abelian group object structure maps $+_B$, 0_B and $-B$ on $f : B \to A$ are λ-ring morphisms.

The proof of these tree statements involve the techniques that we discussed earlier on Witt vectors of modules. The reader is encouraged to do this. Note also, that by construction

$$U_{(A,\lambda_A)} \circ G^\lambda = G \circ U'.$$

Lastly, one has to show that F^λ and G^λ form an adjoint pair compatible with the adjoint pair (F, G), meaning there are unique natural isomorphisms (transformations)

$$G^\lambda \circ F^\lambda \xRightarrow{\varepsilon^\lambda} \mathrm{id} \qquad \text{and} \qquad \mathrm{id} \xRightarrow{\eta^\lambda} F^\lambda \circ G^\lambda$$

such that

$$U_{(A,\lambda_A)}(\varepsilon^\lambda) = \varepsilon \circ U_{(A,\lambda_A)} \text{ and } U'(\eta^\lambda) = \eta \circ U'$$

This means commutativity of the following two diagrams where M is a λ-module, B is the λ-ring $A \ltimes M$ as above, $i : M \to B$ is a chosen embedding of the kernel of $f : B \to A$ into B, of which the first one corresponds to G^λ and the second one corresponds to F^λ.

$$\begin{array}{ccccc}
M & \xrightarrow{\lambda_M} & \mathbb{W}(M) & = & \mathbb{W}(M) \\
\downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle i_{2*}} \\
A \ltimes M & \xrightarrow{\lambda_A \oplus \lambda_M} & \mathbb{W}(A) \ltimes \mathbb{W}(M) & \xrightarrow{i_{1*} + i_{2*}} & \mathbb{W}(A \ltimes)
\end{array}$$

$$\begin{array}{ccccc}
A \ltimes M & \xrightarrow{\lambda_A \oplus \lambda_M} & \mathbb{W}(A) \ltimes \mathbb{W}(M) & \xrightarrow{i_{1*} + i_{2*}} & \mathbb{W}(A \ltimes) \\
\downarrow{\scriptstyle 0_B + i} & & \downarrow{\scriptstyle 0_{B*} + i_*} & & \downarrow{\scriptstyle (0_B + i)_*} \\
B & \xrightarrow{\lambda_B} & \mathbb{W}(B) & = & \mathbb{W}(B)
\end{array}$$

In both diagrams, the left-hand squares commute by naturality and the right-hand squares by the universal property of the direct sum. □

It will be advantageous to be able to work in either category.

We will now define derivations on $\mathscr{M}(A, \lambda_A)$ and bring them together with Beck's more general definition.

Definition 5.31. Let (A, λ_A) be a λ-ring, and (M, λ_M) an (A, λ_A)-module. A derivation

$$D : (A, \lambda_A) \to (M, \lambda_M)$$

is a map of sets such that

1. **Additivity**: for $a, b \in A$, $D(a + b) = D(a) + D(b)$
2. **Leibniz rule**: for $a, b \in A$, $D(ab) = aD(b) + bD(a)$
3. **λ-semilinearity**: for $a \in A$ and $n \in \mathbb{N}$, $\lambda_{M,n}(D(a)) = \sum_{e|n} \lambda_{A,e}(a)^{\frac{n}{e}-1} D(\lambda_{A,e}(a))$

The set of derivations is denoted by $\mathrm{Der}((A, \lambda_A), (M, \lambda_M))$.

Under the equivalence of Prop. 5.30 we have:

Proposition 5.32. *Let (A, λ_A) be a λ-ring, (M, λ_M) and (A, λ_A)-module, and $f : (A \ltimes M, \lambda_{A \ltimes M}) \to (A, \lambda_A)$ the canonical projection. Then there is a bijection*

$$\begin{array}{rcl} \mathrm{Der}((A, \lambda_A), (M, \lambda_M)) & \to & \mathrm{Hom}_{\mathscr{A}_\lambda/(A,\lambda_A)}(\mathrm{id}_{(A,\lambda_A)}, f) \\ D & \mapsto & (\mathrm{id}_A, D) \end{array}$$

Proof. Without λ it is easily verified, that the map from $\mathrm{Der}(A, M)$ to $\mathrm{Hom}_{\mathscr{A}/A}(\mathrm{id}_A, f)$ taking D to (id_A, D) is a bijection.

By abuse of notation, we also write $(\mathrm{id}_A, D) : A \to A \ltimes M$ without the underlying maps. In order to show the claim, we have to show that D is a λ-derivation – meaning, we have to check λ-linearity – iff $(\mathrm{id}_A, D) : A \to A \ltimes M$ is a λ-ring homomorphism, meaning the diagram

$$\begin{array}{ccccc} A & & \xrightarrow{\quad\lambda_A\quad} & & \mathbb{W}(A) \\ \downarrow{\scriptstyle (\mathrm{id}_A, D)} & & & & \downarrow{\scriptstyle (\mathrm{id}_A, D)_*} \\ A \ltimes M & \xrightarrow{\lambda_A \oplus \lambda_M} & \mathbb{W}(A) \ltimes \mathbb{W}(M) & \xrightarrow{i_{1*} + i_{2*}} & \mathbb{W}(A \ltimes M) \end{array}$$

commutes. To see this, let $a \in A$: applying first (id_A, D), then $\lambda_A \oplus \lambda_M$

$$a \mapsto (a, Da) \mapsto (\lambda_A(a), \lambda_M(Da))$$

whose n^{th} Witt component is $(\lambda_{A,n}(a), \lambda_{M,n}(Da))$.

On the other hand, applying first λ_A and then $(\mathrm{id}_A, D)_*$ leads to an element with e^{th} Witt component $(\lambda_{A,e}(a), D\lambda_{M,e}(a))$. Because of Lem. 5.25 and the formula in Lem. 5.26 shows that the diagram commutes if and only if D is λ-linear. □

Recall that classically, K´ahler differentials over a ring A are universal among the derivations over A, in the sense, that for a derivation $D : A \to M$ there is a unique map of A-modules $f : \Omega^1_A \to M$ such that $D = f \circ d$. Another way to express this is by saying the module of K´hler differentials Ω^1_A over A corepresents the functor that assigns to an A-module M the set of derivations $\mathrm{Der}(A, M)$. In the λ-world we have the following analogue.

Lemma 5.33. *Let (A, λ_A) be a λ-ring. There exists a derivation*

$$(A, \lambda_A) \xrightarrow{d} (\Omega^1_{(A,\lambda_A)}, \lambda_{\Omega^1_{(A,\lambda_A)}})$$

which corepresents the functor that to an (A, λ_A)-module (M, λ_M) assignes the set of derivations $\mathrm{Der}\left((A, \lambda_A), (M, \lambda_M)\right)$.

Proof. The target of the map: consider the free (A, λ_A)-module (F, λ_F) generated by the symbols $\{d(a) \mid a \in A\}$, and quotient out the relations that we would like to have: $d(a + b) - d(a) - d(b)$, $d(ab) - bd(a) - ad(b)$ and $\lambda_{F,n}(da) - \sum_{e|n} \lambda_{A,e}(a)^{\frac{n}{e}-1} d\lambda_{A,e}(a)$ for $a, b \in A$, $n \in \mathbb{N}$. The resulting object is denoted $(\Omega^1_{(A,\lambda_A)}, \lambda_{\Omega^1_{(A,\lambda_A)}})$.
The map: d takes a to the class of $d(a)$ under these relations.

By construction, for a λ-derivation $D : (A, \lambda_A) \to (M, \lambda_M)$ there is a unique well-defined map of λ-modules

$$f : (\Omega^1_{(A,\lambda_A)}, \lambda_{\Omega^1_{(A,\lambda_A)}} \to (M, \lambda_M)$$

such that $D = f \circ d$. □

The main theorem of this section identifies Ω^1_A and $\Omega^1_{(A,\lambda_A)}$ as A-modules via the canonical morphism given by the universal property of K´ahler differentials.

Theorem 5.34. *For every λ-ring (A, λ_A) the canonical map*

$$\Omega^1_A \to \Omega^1_{(A,\lambda_A)}$$

is an A-module isomorphism.

Proof. Let

$$(\mathscr{A}/A)_{\mathrm{ab}} \underset{(-)_{\mathrm{ab}}}{\overset{i}{\rightleftarrows}} (\mathscr{A}/A) \qquad \text{and } (\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}} \underset{(-)_{\mathrm{ab}}}{\overset{i^\lambda}{\rightleftarrows}} (\mathscr{A}_\lambda/(A,\lambda_A))$$

be the forgetful functors (forgetting the abelian groups structurem together with their left adjoints. They fit into the following diagram

$$\begin{array}{ccccc}
\mathscr{A}/A & \underset{i}{\overset{(-)_{\mathrm{ab}}}{\rightleftarrows}} & (\mathscr{A}/A)_{\mathrm{ab}} & \underset{G}{\overset{F}{\rightleftarrows}} & \mathscr{M}(A) \\
{\scriptstyle U_{A,\lambda_A}}\uparrow\downarrow{\scriptstyle R_{(A,\lambda_A)}} & & {\scriptstyle U_{(A,\lambda_A)}}\uparrow\downarrow{\scriptstyle R_{(A,\lambda_A)}} & & {\scriptstyle U'}\uparrow\downarrow{\scriptstyle R'} \\
\mathscr{A}_\lambda/(A,\lambda_A) & \underset{i^\lambda}{\overset{(-)_{\mathrm{ab}}}{\rightleftarrows}} & (\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}} & \underset{G^\lambda}{\overset{F^\lambda}{\rightleftarrows}} & \mathscr{M}(A,\lambda_A)
\end{array}$$

where in the right hand square the vertical funtors are adjoint equivalences, as we have seen. (This means that the composition of the top (resp. bottom) adjunctions of the whole square determine the top (resp. bottom) adjunctions of the left-hand square.)

Let $K = i \circ G$. Then we define a functor H, such that it gives rise to an adjunction $(H, K, \varepsilon, \eta)$. Recall what K does: it takes an A-module M to $f : A \ltimes M \to A$ (and the forgets $+_{A\ltimes M}, 0_{A\ltimes M}$ and $-_{A\ltimes M}$). Let H be the functor that assigns to a ring $f : B \to A$ over A the A-module $A \times_B \Omega^1_B$.

Similarly in the λ-world, we define a functor H^λ such that the composition $K^\lambda = i^\lambda \circ G^\lambda$ is its right adjoint: recall that K^λ takes an (A,λ_A)-module (M,λ_M) to the canonical projection $f : (A\ltimes M, \lambda_{A\ltimes M}) \to (A,\lambda_A)$ (and then forgets the abelian group object structure). Define H^λ to be the functor assigning to $f : (B,\lambda_B) \to (A,\lambda_A)$ the (A,λ_A)-module $(A,\lambda_A) \otimes_{(B,\lambda_B)} \Omega^1_{(b,\lambda_B)}$.

Thus we get a diagram of adjunctions

$$\begin{array}{ccc}
\mathscr{A}/A & \underset{K}{\overset{H}{\rightleftarrows}} & \mathscr{M}(A) \\
{\scriptstyle U_{A,\lambda_A}}\uparrow\downarrow{\scriptstyle R_{(A,\lambda_A)}} & & {\scriptstyle U'}\uparrow\downarrow{\scriptstyle R'} \\
\mathscr{A}_\lambda/(A,\lambda_A) & \underset{K^\lambda}{\overset{H^\lambda}{\rightleftarrows}} & \mathscr{M}(A,\lambda_A)
\end{array}$$

with the middle column "missing" from the above diagram. And this shows that up to unique natural isomorphism the composition of functors $R_{(A,\lambda_A)} \circ K$ coincides with the composition $K^\lambda \circ R$. And by uniqueness of the left adjoint, the same holds for the compositions $H \circ U_{(A,\lambda_A)}$ and $U' \circ H^\lambda$.

It follows that the canonical natural transformation

$$A \otimes_B \Omega^1_B \to U'\left((A,\lambda_A) \otimes_{(B,\lambda_B)} \Omega^1_{(B,\lambda_B)}\right)$$

is an isomorphism, and gives the desired result for $(B,\lambda_B) = (A,\lambda_A)$. □

This means, that for a λ-ring (A,λ_A) the A-module of usual differentials Ω^1_A the richer structure of an (A,λ_A)-module. In the case of the λ-ring $(\mathbb{W}(A),\Delta_A)$ this implies the existence of natural F_n-linear maps, that are also denoted $F_n : \Omega^1_{\mathbb{W}(A)} \to \Omega^1_{\mathbb{W}(A)}$.

Theorem 5.35. *Let A be a ring. There are natural F_n-linear maps $F_n : \Omega^1_{\mathbb{W}(A)} \to \Omega^1_{\mathbb{W}(A)}$ such that*

$$F_n(da) = \sum_{e|n} \Delta_{A,e}(a)^{\frac{n}{e}-1} d\Delta_{A,e}(a).$$

Moreover,

1. *for $m,n \in \mathbb{N}$: $F_mF_n = F_{nm}$ and $F_1 = \mathrm{id}$,*
2. *for $n \in \mathbb{N}$ and $a \in \mathbb{W}(A)$: $dF_n(a) = nF_n(da)$,*
3. *for $n \in \mathbb{N}$ and $a \in A$: $F_n(d[a]) = [a]^{n-1}d[a]$.*

Proof. We apply the previous theorem to the λ-ring $\mathbb{W}(A),\Delta_A)$ to get a canonical isomorphism

$$\Omega^1_{\mathbb{W}(A)} \xrightarrow{\sim} \Omega^1_{(\Omega_A,\Delta_A)}.$$

The crucial point is that the target of this map is a $(\mathbb{W}(A), \Delta_A)$-module, which comes together with a map $\lambda_{(\Omega_{\mathbb{W}(A),\Delta_A})}$. We set

$$F_n = \lambda_{(\Omega_{\mathbb{W}(A),\Delta_A}),n} : \Omega^1_{(\mathbb{W}(A),\Delta_A)} \to \Omega^1_{(\mathbb{W}(A),\Delta_A)}$$

as the n^{th} Witt component of this map. It is obviously $F_n = w_n \circ \Delta_A$-linear and by the definition of a λ-derivation satisfies the given formula.

The identities follow with simple calculations. □

5.4 The big de Rham–Witt complex

The theme of the last section of this series is the existence of an initial object in the category of (big) Witt complexes — the big de Rham–Witt complex.

Theorem 5.36. *Let A be a (commutative unital) ring and S a truncation set. There is an initial Witt complex*

$$S \mapsto \mathbb{W}\,\Omega_S(A)$$

over the ring A. Moreover, for each degree q, the canonical map

$$\check{\Omega}^q_{\mathbb{W}_S(A)} \xrightarrow{\eta_S} \mathbb{W}_S\,\Omega^q_A$$

is surjective and we have commutative diagrams

$$\begin{array}{ccc} \check{\Omega}^q_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & \mathbb{W}_S\,\Omega^q_A \\ \downarrow{\scriptstyle R^S_T} & & \downarrow{\scriptstyle R^S_T} \\ \check{\Omega}^q_{\mathbb{W}_T(A)} & \xrightarrow{\eta_T} & \mathbb{W}_T\,\Omega^q_A \end{array} \qquad \begin{array}{ccc} \check{\Omega}^q_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & \mathbb{W}_S\,\Omega^q_A \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d} \\ \check{\Omega}^{q+1}_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & \mathbb{W}_S\,\Omega^{q+1}_A \end{array} \qquad \begin{array}{ccc} \check{\Omega}^q_{\mathbb{W}_S(A)} & \xrightarrow{\eta_S} & \mathbb{W}_S\,\Omega^q_A \\ \downarrow{\scriptstyle F_m} & & \downarrow{\scriptstyle F_m} \\ \check{\Omega}^q_{\mathbb{W}_{\frac{S}{m}}(A)} & \xrightarrow{\eta_{\frac{S}{m}}} & \mathbb{W}_{\frac{S}{m}}\,\Omega^q_A \end{array}$$

The maps on the left hand side in the diagrams from this statement have been defined in Lemma 5.15. It stands to reason to define the complex $\mathbb{W}_S\,\Omega_A$ as quotient of $\check{\Omega}_{\mathbb{W}_S(A)}$ in a way to make the diagrams commute. Furthermore, one defines Verschiebung as maps of graded abelian groups $\mathbb{W}_{\frac{S}{n}}\,\Omega_A \xrightarrow{V_n} \mathbb{W}_S\,\Omega_A$ such that

$$\begin{array}{ccc} \mathbb{W}_{\frac{S}{n}}(A) & \xrightarrow{\eta_{\frac{S}{n}}} & \mathbb{W}_{\frac{S}{n}}\,\Omega^0_A \\ \downarrow{\scriptstyle V_n} & & \downarrow{\scriptstyle V_n} \\ \mathbb{W}_S(A) & \xrightarrow{\eta_S} & \mathbb{W}_S\,\Omega^0_A \end{array}$$

$$\begin{array}{ccc} \mathbb{W}_{\frac{S}{n}}\,\Omega_A & \xrightarrow{V_n} & \mathbb{W}_S\,\Omega_A \\ \downarrow{\scriptstyle R^{\frac{S}{n}}_{\frac{T}{n}}} & & \downarrow{\scriptstyle R^S_T} \\ \mathbb{W}_{\frac{T}{n}}\,\Omega_A & \xrightarrow{V_n} & \mathbb{W}_T\,\Omega_A \end{array}$$

$$\begin{array}{ccccc} & & \mathbb{W}_{\frac{S}{n}}\,\Omega_A \otimes \mathbb{W}_S\,\Omega_A & & \\ & {\scriptstyle \mathrm{id}\otimes F_n}\swarrow & & \searrow{\scriptstyle V_n\otimes \mathrm{id}} & \\ \mathbb{W}_{\frac{S}{n}}\,\Omega_A \otimes \mathbb{W}_{\frac{S}{n}}\,\Omega_A & & & & \mathbb{W}_S\,\Omega_A \otimes \mathbb{W}_S\,\Omega_A \\ \downarrow{\scriptstyle \mu} & & & & \downarrow{\scriptstyle \mu} \\ \mathbb{W}_{\frac{S}{n}}\,\Omega_A & & \xrightarrow{\qquad V_n \qquad} & & \mathbb{W}_S\,\Omega_A \end{array}$$

commute.

The definition of $\mathbb{W}_S\,\Omega_A$ and V_n will be done, as S ranges over all finite truncation sets (which we have seen to suffice), $T \subset S$ over all subtruncation sets, and n over all natural numbers, by induction on the cardinality of S. Then one can show that the object obtained together with this structure actually is a big Witt complex and moreover that it is the initial one.

Proof. To start the induction, let $S=\emptyset$, and define $\mathbb{W}_\emptyset\,\Omega_A$ to be the terminal graded ring which is zero in al degrees, and let

$$\eta_\emptyset : \check{\Omega}_{\mathbb{W}_\emptyset(A)} \to \mathbb{W}_\emptyset\,\Omega_A$$

to be the unique map of graded rings. The maps $R^\emptyset_\emptyset$, F_n, d, and V_n are trivial as well.
Now let S be a finite truncation set, and assume that for all proper truncation sets $T \subsetneq S$, and $U \subset T$ and all $n\in\mathbb{N}$ the maps η_T, R^T_U,F_n, d, and V_n have been defined such that the desired properties are satisfied.

Let N_S be the graded ideal of $\check{\Omega}_{\mathbb{W}_S(A)}$ generated by all sums of the form

$$\sum_\alpha V_n(x_\alpha)dy_{1,\alpha}\cdots dy_{q,\alpha} \quad \text{and} \quad d\left(\sum_\alpha V_n(x_\alpha)dy_{1,\alpha}\cdots dy_{q,\alpha}\right),$$

where $x_\alpha \in \mathbb{W}_{\frac{S}{n}}(A)$ and $y_{1,\alpha},\dots y_{q,\alpha}\in\mathbb{W}_S(A)$ and $n\geqslant 2$, $q\geqslant 1$ such that the projection of the sum

$$\eta_{\frac{S}{n}}\left(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\cdots dy_{q,\alpha}\right)$$

to $\mathbb{W}_{\frac{S}{n}}\,\Omega^q_A$ is zero. Let

$$\mathbb{W}_S\,\Omega_A = \check{\Omega}_{\mathbb{W}_S(A)}/N_S$$

be the quotient, and η_S the quotient map.
Next we define $V_n : \mathbb{W}_{\frac{S}{n}}\,\Omega_A \to \mathbb{W}_S\,\Omega_A$, which has to "commute" with η_S and $\eta_{\frac{S}{n}}$ as map of graded abelian groups by

$$V_n\eta_{\frac{S}{n}}(xF_ndy_1\cdots F_ndy_q) = \eta_S(V_n(x)dy_1\cdots dy_q)$$

which defines V_n uniquely in that every element of $\mathbb{W}_{\frac{S}{n}}\,\Omega^q_A$ can be written as a sum of elements $\eta_{\frac{S}{n}}(XF_ndy_1\cdots dy_q)$ with $x\in\mathbb{W}_{\frac{S}{n}}(A)$ and $y_i\in\mathbb{W}_S(A)$.
We come to the existence and uniqueness of the maps R^S_T, d and F_n, which make the diagrams in the theorem commute. Note that once existence is established, uniqueness is clear due to the commutativity of these diagrams. For the existence, we have to show that applying the left hand vertical maps R^S_T, d and F_n to the q-graded piece of the kernel N^q_S of $\check{\Omega}^q_{\mathbb{W}_S(A)}$ is trivial in the quotient. More precisely, we have to show

$$\begin{aligned}
\eta_T(R^S_T(N^q_S)) &= 0\\
\eta_S(d(N^q_S)) &= 0\\
\eta_{\frac{S}{m}}(F_m(N^q_S)) &= 0
\end{aligned}$$

One has to use the properties established for the maps on $\check{\Omega}$. Let for $n\in\mathbb{N}$

$$\omega = \sum_\alpha V_n(X_\alpha)dy_{1,\alpha}\cdots dy_{q,\alpha}\in\check{\Omega}^q_{\mathbb{W}_S(A)}$$

such that $0=\eta_{\frac{S}{n}}\left(\sum_\alpha x_\alpha F_ndy_{1,\alpha}\dots F_ndy_{q,\alpha}\right)\in\mathbb{W}_{\frac{S}{n}}\,\Omega^q_A$ (this defines a general element of the kernel) and show that

$$\begin{aligned}
\eta_T R^T_S(\omega) &= 0\\
\eta_S(dd\omega) &= 0\\
\eta_{\frac{S}{m}}F_m(\omega) &= 0\\
\eta_{\frac{S}{m}}F_m(d\omega) &= 0
\end{aligned}$$

Rewriting $R^T_S(\omega)$, to show that

$$\eta_T R^T_S(\omega) = \eta_T\left(\sum_\alpha V_n R^{\frac{S}{n}}_{\frac{S}{n}} dR^S_T(y_{1,\alpha})\cdots dR^S_T(y_{q,\alpha})\right)$$

it is enough to show that the following element is zero:

$$\begin{aligned}
\eta_{\frac{T}{n}}\left(\sum_{\alpha} R^{\frac{S}{n}}_{\frac{T}{n}}(x_\alpha) F_n dR^S_T(y_{1,\alpha})\cdots F_n dR^S_T(y_{q,\alpha})\right) &= \eta_{\frac{T}{n}} R^{\frac{S}{n}}_{\frac{T}{n}}\left(\sum_{\alpha} x_\alpha F_n dy_{1,\alpha}\cdots F_n dy_{q,\alpha}\right)\\
&= R^{\frac{S}{n}}_{\frac{T}{n}} \eta_{\frac{S}{n}}\left(\sum_{\alpha} x_\alpha F_n dy_{1,\alpha}\cdots dy_{q,\alpha}\right) \quad \text{by induction hypothesis}\\
&= 0 \quad \text{by induction hypothesis}
\end{aligned}$$

The proofs of the remaining equalities will be left as an exercise.
To complete the definition/construction of $\mathbb{W}_S\,\Omega_A$ together with the maps η_S, R^S_T, d, F_n and V_n, it remains to verify that the three diagrams (two squares and one pentagon) commute.

The diagram

$$\begin{array}{ccc}
\mathbb{W}_{\frac{S}{n}}(A) & \xrightarrow{\eta_{\frac{S}{n}}} & \mathbb{W}_{\frac{S}{n}}\,\Omega^0_A\\
\downarrow V_n & & \downarrow V_n\\
\mathbb{W}_S(A) & \xrightarrow{\eta_S} & \mathbb{W}_S\,\Omega^0_A
\end{array}$$

commutes by definition of the Verschiebung.

The diagram

$$\begin{array}{ccc}
\mathbb{W}_{\frac{S}{n}}\,\Omega_A & \xrightarrow{V_n} & \mathbb{W}_S\,\Omega_A\\
\downarrow R^{\frac{S}{n}}_{\frac{T}{n}} & & \downarrow R^S_T\\
\mathbb{W}_{\frac{T}{n}}\,\Omega_A & \xrightarrow{V_n} & \mathbb{W}_T\,\Omega_A
\end{array}$$

commutes by the following calculation, taking into account that every element of $\mathbb{W}_{\frac{S}{n}}$ can be written as a sum of elements of the form $\eta_{\frac{S}{n}}(xF_n dy_1\cdots dy_q)$ with $x\in\mathbb{W}_{\frac{S}{n}}(A)$ and $y_i\in\mathbb{W}_S(A)$:

$$\begin{aligned}
R^S_T V_n \eta_{\frac{S}{n}}(xF_n dy_1\cdots dy_q) &= R^S_T\eta_S(V_n(x)dy_1\cdots dy_q) \quad \text{by definition of } V_n\\
&= \eta_T R^S_T(V_n(x)dy_1\cdots dy_q) \quad \text{by definition of } R^S_T\\
&= \eta_T(V_n R^{\frac{S}{n}}_{\frac{T}{n}}(x) dR^S_T(y_1)\cdots dR^S_T(y_q)) \quad \text{by induction hypothesis}\\
&= V_n\eta_{\frac{T}{n}}(R^{\frac{S}{n}}_{\frac{T}{n}}(x)F_n dR^S_T(y_1)\cdots F_n dR^S_T(y_q)) \quad \text{by definition of } V_n\\
&= V_n R^{\frac{S}{n}}_{\frac{T}{n}}\eta_{\frac{S}{n}}(xF_n dy_1\cdots dy_q) \quad \text{by definition of } R^{\frac{S}{n}}_{\frac{T}{n}}
\end{aligned}$$

The commutativity of the pentagon is discussed in the exercises.
The next point is to check that what we just defined is indeed a Witt complex over A.As a reminder, for this is needed: $V_1 = \mathrm{id}$, $V_nV_m = V_{nm}$, $F_nV_m = n\,\mathrm{id}$ and $F_mV_n = V_nF_m$ if $(nm) = 1$. The first is clear by definition. For the second identity compute

$$\begin{aligned}
V_{mn}\eta_{\frac{S}{mn}}(xF_{mn}dy_1\cdots F_{mn}dy_q) &= \eta_S(V_{mn}(x)dy_1\cdots dy_q) \quad \text{by definition of } V_{mn}\\
&= \eta_S(V_m(V_n(x))dy_1\cdots dy_q) \quad \text{by the desired equation on } \mathbb{W}(A)\\
&= V_m\eta_{\frac{S}{m}}(V_n(x)F_m dy_1\cdots F_m dy_q) \quad \text{by definition of } V_m\\
&= V_m(V_n(\eta_{\frac{S}{mn}}(x))F_m d\eta_S(y_1)\cdots F_m d\eta_S(y_q)) \quad \text{by existence of } F_m \text{ with } \eta_{\frac{S}{m}}F_m = F_m\eta_S\\
&= V_m(V_n(\eta_{\frac{S}{mn}}(x)F_{mn}d\eta_S(y_1)\cdots F_{mn}d\eta_S(y_q))) \quad \text{by inductive hypothesis}\\
&= V_m(V_n\eta_{\frac{S}{mn}}(xF_{mn}dy_1\cdots F_{mn}dy_q)) \quad \text{by definition of } F_{mn}
\end{aligned}$$

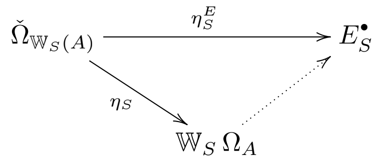
Similarly for the third identity:

$$\begin{aligned} F_n V_n \eta_{\frac{S}{n}}(x F_n dy_1 \cdots dy_q) &= F_n \eta_S (V_n(x) dy_1 \cdots dy_q) \quad \text{by definition of } V_n \\ &= \eta_{\frac{S}{n}} F_n (V_n(x) dy_1 \cdots dy_q) \quad \text{by definition of } F_n \\ &= n \eta_{\frac{S}{n}} (x F_n dy_1 \cdots dy_q) \quad \text{by induction} \end{aligned}$$

The fourth identity will be discussed in the exercises.
Finally, we have to show that the complex which we constructed is initial among Witt complexes over A. To this end, let $E_S^\bullet$ be a Witt complex over A together with the map

$$\eta_S^E : \check{\Omega}_{\mathbb{W}_S(A)} \to E_S^\bullet$$

which was constructed earlier. One has to show that this map factors through $\mathbb{W}_S\,\Omega_A$



Since η_S is by construction surjective, the map f_S has to be unique if it exists. To show existence, by the same reasoning as before, we may assume that the truncation set S is finite, and proceed again by induction on the cardinality of S, the case $S = \emptyset$ being easy, as it is simply the identity. Thus let S be a finite truncation set, and assume that for every proper subtruncation set $T \subsetneq S$, the factorisation $\eta_T^E = f_T \eta_T$ exists. The proceeding is now similar to the existence of the maps R_T^S, F_n, d, as we have to show again, that for any $n \in \mathbb{N}$, $x_\alpha \in \mathbb{W}_{\frac{S}{n}}(A)$ and $y_{1,\alpha}, \ldots, y_{y,\alpha} \in \mathbb{W}_S(A)$ such that

$$\eta_{\frac{S}{n}} \left(\sum_\alpha x_\alpha F_n dy_{1,\alpha} \cdots F_n dy_{q,\alpha} \right) \in \mathbb{W}_{\frac{S}{n}}\,\Omega_A^q$$

vanishes, the element

$$\eta_S^E \left(\sum_\alpha V_n(x_\alpha) dy_{1,\alpha} \cdots dy_{q,\alpha} \right) \in E_S^q$$

vanishes as well.

Using that $E_S^\bullet$ is a Witt complex, we find (with some intermediate steps that are omitted) with the inductive hypothesis that

$$\eta_S^E \left(\sum_\alpha V_n(x_\alpha) dy_{1,\alpha} \cdots dy_{q,\alpha} \right) = V_n f_{\frac{S}{n}} \eta_{\frac{S}{n}} \left(\sum_\alpha x_\alpha F_n dy_{1,\alpha} \cdots F_n dy_{q,\alpha} \right)$$

which vanishes by induction.

This is the induction step to get the factorisation for S.

FInally, one has to show that the so obtained maps f_S for varying S constitute a map of Witt complexes, which means that it commutes with the respective d's, F_n's and V_n's. We have seen in Corollary 5.16 that the maps η^E commute with Frobenius, more precisely for $m \in \mathbb{N}$

$$F_m \circ \eta_S^E = \eta_{\frac{S}{m}} \circ F_m$$

and by construction, the same holds true for the maps η in $\mathbb{W}\,\Omega$. It follows that

$$F_m \circ f_S = f_{\frac{S}{m}} \circ F_m$$

for all $m \in \mathbb{N}$. Likewise, since η and η^E commute with the differentials d, the maps f_S are bound to do so as well. Finally, it remains to show that for every truncation set S and for every positive integer m, one has $f_S \circ V_m = V_m \circ f_{\frac{S}{m}}$: again by the reasoning that every element of $\mathbb{W}_{\frac{S}{m}}$ can be written as a sum of elements of the form $\eta_{\frac{S}{m}}(xF_n dy_1 \cdots dy_q)$ with $x \in \mathbb{W}_{\frac{S}{m}}(A)$ and $y_i \in \mathbb{W}_S(A)$:

$$\begin{aligned}
f_S V_m \eta_{\frac{S}{m}}(x \cdot F_m dy_1 \cdots F_m dy_q) &= f_S \eta_S(V_m(x) \cdot dy_1 \cdots dy_q) \quad \text{by definition of } V_n \\
&= \eta_S^E(V_m(x) \cdot dy_1 \cdots dy_q) \quad \text{by factorisation of } \eta^E \\
&= \eta_S^E(V_m(x)) \cdot \eta_S^E(dy_1 \cdot dy_q) \quad \text{by multiplicativity of } \eta^E \\
&= V_m(\eta^E_{\frac{S}{m}}(x)) \cdot \eta_S^E(dy_1 \cdots dy_q) \quad \text{since } V_m \text{ and } \eta^E \text{ commute in degree zero} \\
&= V_m(\eta^E_{\frac{S}{m}}(x) \cdot F_m \eta_S^E(dy_1 \cdots dy_m)) \quad \text{by definition} \\
&= V_m(\eta^E_{\frac{S}{m}}(x) \cdot \eta^E_{\frac{S}{m}} F_m(dy_1 \cdots dy_m)) \quad \text{since } \eta^E \text{ and } F_m \text{ commute} \\
&= V_m(\eta^E_{\frac{S}{m}}(x \cdot F_m dy_1 \cdots dy_m) \quad \text{by multiplicativity of } \eta^E \\
&= V_m f_{\frac{S}{m}} \eta_{\frac{S}{m}}(x \cdot F_m dy_1 \cdots F_m dy_q) \quad \text{by factorisation of } \eta^E
\end{aligned}$$

This completes the proof of the theorem. □

Definition 5.37. The initial Witt complex $\mathbb{W}_S \Omega_A$ is called the big de Rham–Witt complex for the truncation set S of A. If $S = \mathbb{N}$, it is denotes by $\mathbb{W}\Omega_A$ and called the big de Rham–Witt complex of A.

It is clear by definition, that considering the unit truncation set, one obtains the usual de Rham complex. More precisely,the map

$$\eta_{\{1\}} : \Omega^q_A \xrightarrow{\sim} \mathbb{W}_{\{1\}} \Omega^q_A$$

is an isomorphism for all q. Moreover, in degree zero, one has an isomorphism

$$\eta_S : \mathbb{W}_S(A) \to \mathbb{W}_S \Omega^0_A$$

for all truncation sets S. This is in line with the p-typical de Rham–Witt complex.

It is possible to define a relative version of the big de Rham–Witt complex, using relative λ-derivations. This is a big version of Langer and Zink's relative de Rham–Witt complex [6].

References

[1] Jonathan Mock Beck. Triples, algebras and cohomology. *Repr. Theory Apl. Categ.*, 2:1–59, 2003.

[2] James Borger. Lambda-rings and the field with one element. arXive:0906.3146, 2009.

[3] James Borger. The basic geometry of witt vectors i: The affine case. *Algebra Number Theory*, 5:231–285, 2011.

[4] Lars Hesselholt. The big de Rham–Witt complex. *Acta Mathematica*, 214:135–207, 2015.

[5] Lars Hesselholt and Ib Madsen. On the de Rham–Witt complex in mixed characteristic. *Ann. Sci. Ecole Norm. Sup.*, 37:1–43, 2004.

[6] Andreas Langer and Thomas Zink. De Rham–Witt cohomology for a proper and smooth morphism. *Journal of the Inst. of Math. Jussieu*, 3(2):231–314, 2004.

[7] Wilberd van der Kallen. Descent for the k-theory of polynomial rings. *Mathematische Zeitschrift*, 191:405–416, 1986.

[8] Ernst Witt. Zyklische körper und algebren der charakteristik p vom grad pn. struktur diskret bewerteter perfekter körper mit vollkommenem restklassenkörper der charakteristik p. *Journal für die reine und angewandte Mathematik*, 176:126–140, 1937.

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