3 Crystalline cohomology

As we have mentioned, one of the objectives to construct a de Rham–Witt complex was to be able to compute crystalline cohomology more explicitly. In this section, we want to give a quick review of the basic concepts of crystalline cohomology. The standard reference for crystalline cohomology is of course Pierre Berthelot and Arthur Ogus' book [1]. A ver quick and to the point overview can be found in Antoine Chambert-Loir's survey article [2] and in Luc Illusie's paper [3].

3.1 Divided powers

The idea of crystalline cohomology goes back, as so many concepts in algebraic geometry, to Grothendieck. It was clear, at a very early stage of the idea, that so called divided powers would be needed for the construction, as it basically concerns an integration process.

Definition 3.1. Let A be a ring and $I \subset A$ an ideal. A PD-structure on I is a sequence of maps $\gamma_n : I \to A$ such that

— $\gamma_0(x) = 1$ and $\gamma_1(x) = x$ for all $x \in I$
— $\gamma_n(x) \in I$ for $n \geqslant 1$ and $x \in I$
— $\gamma_n(x+y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y)$ for all $x, y \in I$
— $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$ for all $\lambda \in A$ and $x \in I$
— $\gamma_n(x)\gamma_m(x) = \binom{m+n}{n}\gamma_{m+n}(x)$ for all $x \in I$ and $m, n \in \mathbb{N}$
— $\gamma_m(\gamma_n(x)) = \frac{(mn)!}{m!(n!)^m}\gamma_{mn}(x)$ for all $x \in I$ and $m, n \in \mathbb{N}$

In this case, we say that A is a PD-ring.

Where do these formulae come from? They ensure that morally "$\gamma_n(x) = \frac{x^n}{n!}$". These elements are needed to integrate — which should be clear if we just recall basic formulae from Calculus.

Examples 3.2.
1. For a perfect ring A of characteristic $p > 0$, the ideal (p) in the ring of Witt vectors $W(A)$ has a natural PD-structure, given by $\gamma_n(p) = \frac{p^n}{n!}$ which makes sense, sind the p-adic valuation of $\frac{p^n}{n!}$ is positive for all $n \in \mathbb{N}_0$ and strictly positive for $n \geqslant 1$.
2. For any ring A, we define an A-PD-algebra in n variables

$$A\langle x_1, \dots x_n\rangle = \bigoplus_{r \geqslant 0} \Gamma^r$$

where a base of Γ^r as A-modules is given by symbols $x_1^{[k_1]} \cdots x_n^{[k_n]}$ such that $k_1 + \dots k_n = r$, $k_i \in \mathbb{N}_0$. The algebra structure is given by the relations $x_i^{[m]} x_i^{[n]} = \binom{m+n}{n} x_i^{[m+n]}$. The ideal $I = A^+\langle x_1, \dots, x_n\rangle = \bigoplus_{r \geqslant 1} \Gamma^r$ then has a unique PD-structure such that $\gamma_r(x_i) = x_i^{[r]}$.

Remark 3.3. Note that if A is annihilated by a $n \geqslant 2$, then a PD ideal $I \subset A$ is automatically a nil-ideal, since $x^n = n!\gamma_n(x) = 0$ for every $x \in I$. In particular $\operatorname{Spec} A$ and $\operatorname{Spec} A/I$ have the same underlying topological space.

The idea behind crystalline cohomology is to locally compute de Rham-type complexes with additional PD-structure. Let's take the non-PD setting as a model:

Let $\mathscr{T}$ be a topos and A a (commutative unital) ring of $\mathscr{T}$.

Definition 3.4. We call an anticommutative graded A-algebra B, in positive degrees, with an A-linear differential $d : B^i \to B^{i+1}$ such that $d^2 = 0$ and $d(xy) = (dx)y + (-1)^i x dy$, a differential graded A-algebra B. A morphism of differential graded A-algebras is a morphism of A-algebras compatible with the differential structures.

Recall that for an A-algebra R the de Rham complex $\Omega_{R/A}$ is universal in the sense that for any A-dga B, every A-algebra morphism $R \to B^0$ extends in a unique way to an A-dga morphism $\Omega_{R/A} \to B$.

Proposition 3.5. *Let A be as above and denote by $\mathcal{dga}^{\geqslant 0}(A)$ the category of differential graded A algebras. The functor*

$$\mathcal{Alg}(A) \to \mathcal{dga}^{\geqslant 0}(A)\ ,\ C \mapsto \Omega_{C/A}$$

is left adjoint to the forgetful functor

$$\mathcal{dga}^{\geqslant 0}(A) \to \mathcal{Alg}(A)\ ,\ B \to B^0.$$

We also say, the object $\Omega_{C/A}$ is initial in the category $dga^{\geqslant 0}(A)$.

Definition 3.6. Let B be an A-dga. A differential graded B-module (or B-dgm) is a graded B-module M together with a differential $d : M^i \to M^{i+1}$ such that $d^2 = 0$ and $d(bx) = (db)x + (-1)^i bdx$ for $b \in B^i$ and $x \in M^j$. A morphism of B-dgm's is a morphism of B-modules compatible with the differential structure. We can define left and right B-dga's. Every right B-dgm can be seen as a left B-dgm via the anti-commutative law $bx = (-1)^{ij} xb$.A differential graded ideal (dgi) of B is a sub B-dgm of B.

If $I^0 \subset B^0$ is an ideal, then the ideal in B generated by I^0 and dI^0 is a dgi of B with zero component I^0, and it's the smallest dgi with this property (it is in fact the dgi generated by I^0). Furthermore, for $n \in \mathbb{N}$, I^n is generated additively by elements of the form $bdx_1 \cdots dx_n$ with $b \in B^0$ and $x_i \in I^0$. If I is a B-dgi, B/I is an A-dga.

Definition 3.7. Let E be a B^0-module. A connection on E with respect to B is a morphism

$$\nabla : E \to E \otimes_{B^0} B^1$$

such that $\nabla(bx) = b\nabla x + x \otimes db$.

Every connection ∇ extends in a unique way to a morphism $\nabla : E \otimes_{B^0} B^i \to E \otimes_{B^0} B^{i+1}$ such that $\nabla(b \otimes x) = b\nabla x + x \otimes db$ for $b \in B^i$ and $x \in E$.

Definition 3.8. We say that ∇ is integrable if $\nabla^2 = 0$. If this is the case, $(E \otimes B, \nabla)$ is a B-dgm

We want to take this idea to the PD-world.

Definition 3.9. Let (B, I, γ) ba an A-PD-algebra. The ideal of $\Omega_{B/A}$ generated by the elements $d(\gamma_n(x)) - \gamma_{n-1}(x)dx$ for $x \in I$ is a dgi J. Thus the quotient

$$\Omega_{B/A,\gamma} := \Omega_{B/A}/J$$

is an A-dga called the PD-de Rham complex of B/A.

It is the initial object in the category of PD-A-dga's: if C is an A-dga with a PD-ideal K of C^0 and PD-structure δ compatible with d in the sense that $d(\delta_n x) = \delta_{n-1}(x)dx$, then any morphism of A-PD-algebras $f^0 : B \to C^0$ extends uniquely to a homomorphism of A-dga's $f : \Omega_{B/A,\gamma} \to C$. Now let (A, I, γ) be a PD-ring in $\mathscr{T}$, B an A-algebra, $J \subset B$ an ideal. Let $\overline{B} = D_{B,\gamma}(J)$ be the decided power envelope of (B, J) with respect to γ (this is $B\langle J\rangle$ from the example above modes out by relations, that make the PD-structure compatible with γ). Denote by $\overline{J}$ the the associated PD-ideal. $\overline{B}$ is generated as B-algebra by the divided powers $x^{[n]}$, for $x \in J$.

Proposition 3.10. *The derivation $d : B \to \Omega^1_{B/A}$ extends in a unique way to a derivation $d : \overline{B} \to \overline{B}\Omega^1_{B/A}$ such that*

$$dx^{[n]} = x^{[n-1]} \otimes dx,$$

for $x \in J$ and $n \in \mathbb{N}$.

In [1] this comes out of the theory of hyper PD-stratifications, but it can also be verified directly.

The derivation $d : \overline{B} \to \overline{B} \otimes_B \Omega^1_{B/A}$ then extends uniquely to $\overline{B} \otimes_B \Omega_{B/A}$ and $d^2 = 0$. The universality of the A-dga $\Omega_{\overline{B},A,[-]}$ shows that there is a unique homomorphism

$$\Omega_{\overline{B},A,[-]} \to \overline{B} \otimes_B \Omega_{B/A} \tag{3.1}$$

which is the identity in degree zero.

Proposition 3.11. *The homomorphism (3.1) is an isomorphism.*

Proof. The homomorphism of grade A-aglebras

$$\overline{B} \otimes_B \Omega_{B/A} \to \Omega_{\overline{B}/A,[-]}$$

which is the identity in degree zero and given by the composition

$$\overline{B} \otimes_B \Omega^1_{B/A} \to \Omega^1_{\overline{B}/A} \to \Omega^1_{\overline{B}/A,[-]}$$

is compatible with the differential and therefore an inverse of the morphism in question. ☐

3.2 Crystalline site and crystalline cohomology

Let S be a scheme such that p is locally nilpotent, I a quasi-coherent ideal of $\mathscr{O}_S$, and γ a PD-structure on I — in other words (S, I, γ) is a PD-scheme. Think of $S = W_n(S_0)$ for S_0 the Spec of a perfect field. Let X be an S-scheme such that γ extends to a PD-structure on X. We will define the crystalline site of X with respect to (S, I, γ). The objects are S-PD-thickenings of Zariski open subsets of X.

The crystalline site of X over S is denoted by $\mathrm{Cris}(X/S)$.

— The objects are triples (U, T, δ), where U is a Zariski open of X, T is an S scheme together with a closed immersion $U \hookrightarrow T$ given by an ideal J with PD-structure δ compatible with γ (thus J is a nil-ideal and U and T have the same underlying topological space).

— The morphisms are morphisms of triple $(U, T, \delta) \to (U', T', \delta')$ sending $U \to U'$ and $T \to T'$ compatible with the PD-structure.

— The covering families are $(U_\alpha, T_\alpha, \delta_\alpha) \to (U, T, \delta)$ such that the T_α cover T.

The associated tops is denoted by $(X/S)_{\mathrm{cris}}$. One can describe a sheaf $\mathscr{E}$ on the crystalline site explicitly, by giving for each (U, T, δ) a sheaf $\mathscr{E}_{(U,T,\delta)}$ on T for the Zariski topology, and for each map $f : (U', T', \delta') \to (U, T, \delta)$ a transition map $f^* \mathscr{E}_{(U,T,\delta)} \to \mathscr{E}_{(U',T',\delta')}$ which satisfies transitivity and is an isomorphism if $T' \to T$ is an open immersion. A useful feature of this interpretation is, that the Zariski site has enough points, which means, that we can check if a map of sheaves $\nu : F \to G$ is an isomorphism, by looking at stalks: It is enough to check that for each $x \in X$ and each S-PD-thickening T of a Zariski neighbourhood of x, $(F_T)_x \to (G_T)_x$ is an isomorphism.

Examples 3.12. The structure sheaf $\mathscr{O}_{X/S}$ is given by the cofunctor $(U, T, \delta) \mapsto \mathscr{O}_T$. But also the cofunctor $(U, T, \delta) \mapsto \mathscr{O}_U$ defines a sheaf of rings denoted by $\mathscr{O}_X$. And the PD-ideal sheaf $\mathscr{J}_{X/S} \subset \mathscr{O}_{X/S}$ that associated to (U, T, δ) the defining ideal of the closed immersion $U \hookrightarrow T$, $(U, T, \delta) \mapsto \mathrm{Ker}(\mathscr{O}_T \to \mathscr{O}_U)$. In fact, there is a short exact sequence

$$0 \to \mathscr{J}_{X/S} \to \mathscr{O}_{X/S} \to \mathscr{O}_X \to 0.$$

Definition 3.13. A sheaf of $\mathscr{O}_{X/S}$-modules is a crystal if all the transition morphisms are isomorphisms.

It is preferable to work with the crystalline topos as opposed to the crystalline site, because one has more functoriality: one has for example inverse image sheaves. But this needs some checking and abstract nonsense.

Example 3.14. An example to keep in mind is that of a scheme X over a perfect field K of characteristic $p > 0$, and $S = W_n(k)$ with the canonical PD-structure. Then the objects of $\mathrm{Cris}(X/W_n)$ are given by diagrams

$$\begin{array}{ccc} U & \hookrightarrow & T \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, k & \longrightarrow & \mathrm{Spec}\, W_n \end{array}$$

such that the ideal $\mathrm{Ker}(\mathscr{O}_T \to \mathscr{O}_U)$ has a PD-structure compatible with the canonical Witt vector PD-structure.

To define the global section functor recall that for a topos $\mathscr{T}$ and $T \in \mathscr{T}$, $\Gamma(T, -)$ is the functor $F \mapsto \mathrm{Hom}_{\mathscr{T}}(F, T)$. If e is the final object in $\mathscr{T}$, we write $\Gamma(e, F) =: \Gamma(\mathscr{T}, F) =: \Gamma(F)$. The final object for a topos is the sheafification of the constant pre sheaf given by $\{0\}$ on each U. For an ordinary topological space X this sheaf is represented by the open subset X of X itself. In case of the crystalline topos, it is not representable however. In general, a section $s \in \Gamma(\mathscr{T}, F) = \mathrm{Hom}(e, F)$ is a compatible collection of sections $s_T \in F(T)$ for every $T \in X$, i.e. an element in $\varprojlim_{T \in X} F(T)$.

Let X_{Zar} be the Zariski topos of X. Then there is a canonical projection

$$u_{X/S} : (X/S)_{\mathrm{cris}} \to X_{\mathrm{Zar}}$$

given by

$$\begin{aligned} u_{X/S*} : &\quad \Gamma(U, u_{X/S,*}\mathscr{E}) = \Gamma((U/S)_{\mathrm{cris}}, \mathscr{E}) \\ u_{X/S}^{-1} : &\quad (u_{X/S}^{-1}(\mathscr{F}))_{(U,T,\delta)} = \mathscr{F}\big|_U \end{aligned}$$

It is clear, that $u_{X/S}^{-1}$ commutes with arbitrary inverse limits, so that we really have a morphism of topoi, but not of ringed topoi. It is a morphism of ringed topoi if X is considered with the sheaf $f^{-1}\mathscr{O}_S$ (for $f: X \to S$). If $f_{\text{cris}}: (X/S)_{\text{cris}}? \to S$ is the projection, then there is a canonical isomorphism in the derived category

$$Rf_{\text{cris}}\mathscr{E} = Rf_* Ru_{X/S*}\mathscr{E}$$

In particular, $R\Gamma(X_{\text{Zar}}, Ru_*\mathscr{E}) \cong R\Gamma((X/S)_{\text{cris}}, \mathscr{E})$.

Recall now the calculus of $(X/S)_{\text{cris}}$ in case there is a closed immersion $j : X \to Z$ into a smooth scheme. In general the ideal $\text{Ker}(\mathscr{O}_Z \to \mathscr{O}_X)$ does not have divided powers,thus we consider the PD-envelope$\overline{Z}$ of X in Z, meaning, that we formally add divided powers to the defining ideal in a universal way, and obtain $X \hookrightarrow \overline{Z} \to Z$. Moreover for a crystal $\mathscr{E}$ there is a unique integrable connection

$$d : \mathscr{E}_{\overline{Z}} \to \mathscr{E}_{\overline{Z}} \otimes \Omega^1_{Z/S}$$

compatible with the PD-structure. If $\mathscr{E} = \mathscr{O}_{X/S}$ this gives just the complex $\mathscr{O}_{\overline{Z}} \otimes \Omega_{Z/S} = \Omega_{\overline{Z}/S,[-]}$. A fundamental theorem of Berthelot and Grothendieck says:

Theorem 3.15. *There is a canonical isomorphism*

$$Ru_{X/S*}\mathscr{E} \xrightarrow{\sim} \mathscr{E}_{\overline{Z}} \otimes \Omega_{Z/S}.$$

In particular, for $\mathscr{E} = \mathscr{O}_{X/S}$ this isomorphism is compatible with the natural product structures on both sides. The proof uses a simplicial complex called the Čech-Alexander complex and the so-called crystalline Poincaré lemma. Even if globally X is not smoothable, it is locally, and using cohomological descent, we can treat this case as well.

Lemma 3.16. *Let A be a ring. The de Rham complex of $A[t_1, \dots, t_n]$ with coefficients in $A\langle t_1, \dots, t_n\rangle$ (with the integrable connection $t_i^{[k]} \mapsto t_i^{[k-1]} dt_i$) is a resolution of A.*

Now let $S = W_n$. If X has a smooth lift over W_n, crystalline cohomology of X corresponds to the de Rham cohomology of the oft.

Corollary 3.17. *If Z/W_n is a smooth lift of X, then $\overline{Z} = Z$ and*

$$H^*_{cris}(X/Wn) = H^*_{dR}(Z/W_n).$$

The isomorphism of Theorem 3.15 is functorial in X and compatible with base change of (S, I, γ). In particular, let X/k and $S = W_n$ with Frobenius σ. Then the absolute Frobenius of X, $\mathrm{F} : X \to X$ induces a σ-linear morphism in cohomology

$$\mathrm{F} : H^*(X/W_n) \to H^*(X/W_n).$$

References

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