Lectures on the de Rham–Witt complex

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Abstract

The (classical p-typical) de Rham-Witt complex is a complex of sheaves on a scheme over a perfect field of prime characteristic p.More precisely, it is a pro-system of differential graded algebras. In degree zero, it gives the Witt vectors and the first complex in the inverse limit is the de Rham complex. It provides an explicit way to compute crystalline cohomology. The constructions go back to Bloch,Deligne and Illusie. Since then various extensions and different methods are available. Current developpements have applications in K-theory and p-adic Hodge theory.

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1 Introduction

The de Rham–Witt complex plays an important role in arithmetic geometry, it occurs in different forms and shapes at different places. Historically, the p-typical de Rham-Witt complex as we know it nowadays goes back to Illusie. Why would one mix the concpet of de Rham complex with Witt vectors in the first place? Chambert-Loir in his survey [5] gives (at least) two reasons:

— To have a concrete and intrinsic way to compute crystalline cohomology of a scheme in characterisitic $p \neq 0$ (which is a characteristic 0 object), one would like to have some sort of complex with similar properties as the de Rham complex. The de Rham complex itself does not work, so one need some sort of modification. In particular, it turns out that one needs "divided powers" - which naturally occur in the ring of Witt vectors.

— It was also hoped that such a complex would allow to compare crystalline cohomology to other cohomology theory. Is there for example an analogue of the Hodge to de Rham spectral sequence, can one relate the Serre cohomology $H^*(X, W\mathscr{O})$ or étale cohomology $H^*_{\text{ét}}(X \otimes \overline{k}, \mathbb{Z}_p)$ to crystalline cohomology,?

Bloch was the first who gave a construction a such complex using K-theory in order to answer these questions. However, it was restricted to small enough dimensions and primes $p \neq 2$. Deligne later suggested a construction using differential calculus, which was then carried out by Illusie, and Illusie –Raynaud.

The de Rham–Witt complex as defined by Illusie is a complex of sheaves on a scheme over a perfect field k of characteristic $p \neq 0$. There are generalissations of this to $\mathbb{Z}_{(p)}$-schemes by Langer and Zink [10] – the relative de Rham–Witt complex – and by Hesselholt and Madsen [8] resepctively – the absolute de Rham–WItt complex.

Even further goes the big de Rham–Witt complex due to Hesselholt and Madsen. It is a multi-prime version of the de Rham–Witt complex which is closely related to homological algebra, as it was introduced with the purpose of giving an algebraic description of the equivariant homotopy groups in low degrees of Bökstedt's topological Hochschild spectrum of a commutative ring.This functorial algebraic description, in turn, is essential for understading algebraic K-theory by means of the cyclotomic trace map of Bökstedt–Hsiang–Madsen [7]. There is an improvement of this construction due to Lars Hesselholt using the theory of λ-rings.

If there is interest, it is possible to discuss this more detailed later on in the course.

2 Witt vectors

Witt vectors have originally been developed by Ernst Witt [14] as a generalisation of the p-adic numbers. The p-typical version often occurs in mixed characteristic and lifting problems, providing a construction of the unramified extension of the p-adic integer. They are equipped with different universal properties, depending on which view point is to be taken. Furthermore, there is the generalisation to big Witt vectors, from which the p-typical ones for every prime p can be deduced.

2.1 Strict p-rings with perfect residue rings

Much of this follows [12] and [11].

Definition 2.1. Let W be a ring and A perfect of characteristic $p > 0$. Then W is a p-ring with residue ring A if there is $\pi \in W$ such that W is separated for the π-adic topology and complete, and $A = W/\pi$.

In particular $p \in \pi W$. A p-ring always has a unique set of multiplicative representatives $[-] : A \to W$, and for a sequence of elements $\{a_i \in A\}_{i \in \mathbb{N}}$ the series

$$\sum_{i \in \mathbb{N}_0} [a_i] p^i \tag{2.1}$$

converges to an element in W.

Definition 2.2. The ring W is said to be strict if $p = \pi$.

In this case every element $a \in W$ can be written in a unique way in the form (2.1), and the a_i are called coefficients of a.

Example 2.3. Let $S = \mathbb{Z}[X_i^{p^{-\infty}}, i \in \mathbb{N}_0]$ Its p-adic completion $\widehat{S} = \mathbb{Z}_p[X_i^{p^{-\infty}}, i \in \mathbb{N}_0]$ is a strict p-ring with residue ring $\mathbb{F}_p[X_i^{p^{-\infty}}, i \in \mathbb{N}_0]$, which is perfect of characteristic $p \neq 0$. The variables X_i are multiplicative representatives in $\widehat{S}$ because they have $p^{n\text{th}}$ roots for each $n \geqslant 0$. (In fact, the multiplicative system of representatives is characterised by the fact, that the elements are $(p^n)^{\text{th}}$ roots for all n.) This ring will be useful in a later proof.

We look at the particular case, that A is a perfect ring of characteristic p. In this case, we have the following theorem.

Theorem 2.4. *There is up to unique isomorphism a unique strict p-ring denoted by $W(A)$, called the ring of Witt vectors with coefficients in A, with residue ring A. Moreover on has:*

1. *There is a unique system of representatives $[-] : A \to W(A)$, called Teichmüller representatives, and this map is multiplicative*
$$[ab] = [a][b].$$
2. *Each element $a \in W(A)$ has a unique representation as a sum*
$$\underline{a} = \sum_{n=0}^{-\infty} [a_n] p^n$$
with $a_n \in A$.
3. *The construction of $W(A)$ and $[-]$ is functorial in A, i.e. for a homomorphism $f : A \to A'$ of perfect rings of characteristic p, there is a unique homomorphism $W(f) : W(A) \to W(A')$ such that the diagrams*
$$\begin{tikzcd} W(A) \arrow[r, "\tilde{f}"] \arrow[d] & W(A') \arrow[d] \\ A \arrow[r, "f"] & A' \end{tikzcd}$$
and
$$\begin{tikzcd} W(A) \arrow[r, "\tilde{f}"] & W(A') \\ A \arrow[r, "f"] \arrow[u, "{[-]}"] & A' \arrow[u, "{[-]}"'] \end{tikzcd}$$

commute.

Example 2.5. Any unramified extension $R/\mathbb{Z}_p$ with residue field $k = R/p \cong \mathbb{F}_q$, for some $q = p^r$ is a strict p-ring, and hence according to the theorem, the unique strict p-ring with residue field $\mathbb{F}_q$. The Teichmüller representatives have a very nice description. As $\mathbb{F}_q^* \cong \mathbb{Z}/(q-1)$, the non-zero elements of $\mathbb{F}_q$ are the roots of the polynomial $x^{q-1} - 1$. By Hensel's Lemma, each $x \in \mathbb{F}_q$ has a lift $[x] \in R$ such that also $[x]^{q-1} - 1 = 0$ in R. Lastly, we set $[0] = 0 \in R$. This set, the $(q-1)$st roots of unity togehter with 0 is of course multiplicative, and by the theorem this gives exactly the Teichmüller representatives of R.

There is a rather non-constructive proof of the existence and uniqueness of $W(A)$.

Consider the ring $\widehat{S} = \mathbb{Z}_p[X_i^{p^{-\infty}}, Y_j^{p^{-\infty}} \, : \, i,j \in \mathbb{N}_0]$, and take the elements
$$x = \sum [X_i] p^i \quad , \quad y = \sum [Y_i] p^i.$$
Then for any operation $* = +, -, \cdot$, the composition $x * y$ is again an element in $\widehat{S}$, and thus can be written again in the form
$$x * y = \sum [Q_i^*] p^i \quad , \quad \text{with } Q_i^* \in \mathbb{F}_p[X_i^{p^{-\infty}}, Y_j^{p^{-\infty}} \, : \, i,j \in \mathbb{N}_0].$$
As the Q_i^* are polynomials with coefficients in the prime field $\mathbb{F}_p$ we can evaluate them in any perfect ring of characteristic p, and this allows us to determine the structure of a strict p-ring.

Proposition 2.6. *Let W be a p-ring with residue ring A. Let a_i and $b_j \in A$. Then*
$$\sum [a_i] p^i * \sum [b_i] p^i = \sum [c_i] p^i$$
with $c_i = Q_i^*(a_0, \ldots, b_o, \ldots)$.

Proof. There is a homomorphism $\theta : \mathbb{Z}[X_i^{p^{-\infty}}, Y_j^{p^{-\infty}} \, : \, i,j \in \mathbb{N}_0] \to W$ sending $X_i \mapsto [a_i]$, which extends by continuity to $\mathbb{Z}_p[X_i^{p^{-\infty}}, Y_j^{p^{-\infty}} \, : \, i,j \in \mathbb{N}_0]$ and induces a morphism on residue fields
$$\overline{\theta} : \mathbb{F}_p[X_i^{p^{-\infty}}, Y_j^{p^{-\infty}} \, : \, i,j \in \mathbb{N}_0] \to A$$

sending the $X_i \mapsto a_i$ and $Y_i \mapsto b_i$. As θ is a morphism of p-rings, it commutes with multiplicative representatives, and we obtain

$$\begin{aligned}\sum [a_i]p^i * \sum [b_i]p^i &= \theta(x) * \theta(y) = \theta(x * y) \\ &= \sum \theta([Q_i^*])p^i \\ &= \sum [\overline{\theta}(Q_i^*)]p^i\end{aligned}$$

and $\overline{\theta}(Q_i^*) = c_i$. □

Proposition 2.7. *Let W and W' be p-rings, with residue rings A and A', and assume further that W is strict. For any homomorphism $f : A \to A'$ there is a unique homomorphism $g : W \to W'$, such that the diagram*

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d] & W \arrow[d] \\ A \arrow[r, "f"] & A \end{tikzcd}$$

is commutative.

Proof. We have already mentioned that a morphism of p-rings always commutes with the system of multiplicative representatives. For an element $a \in W$ with coordinates $\{\alpha_i \in A\}_i$ one should have

$$g(a) = \sum_{i=0}^{-\infty} g([\alpha_i]_W)p^i = \sum_{i=0}^{-\infty} [f(\alpha_i)]_{W'}.$$

Because W is strict, the α_i determine a uniquely, so the above expression shows the uniquenes of g if it exists. In fact, one can take this expression as definition to get existence, if we remark, that it defines in fact a homomorphism of rings, commuting with multiplication, addition and subtraction by Proposition 2.6. □

Corollary 2.8. *Two strict p-rings with the same residue ring are canonically isomorphic.*

Lemma 2.9. *Let $f : A \to A'$ a surjective homomorphism of perfect rings of characterisitic p. If there exists a strict p-ring W with residue ring A, there exists as well a strict p-ring W' with residue ring A'.*

Proof. We will define W' as quotient of W. For this, we consider an equivalence relation: Let a and $b \in W$ with coordinates $\{\alpha_i \in A\}_i$ and $\{\beta_i \in A\}_i$. Then $a \equiv b$ if $f(\alpha_i) = f(\beta_i)$ for all $i \in \mathbb{N}_0$. If $a \equiv a'$ and $b \equiv b'$, one shows using Proposition 2.6, that $a * b \equiv a' * b'$ for $* = +, -, \cdot$. Thus the quotient of W by this equivalence relation

$$W' := W/\sim$$

is a ring.

Let $x \in W'$ be in the immage of an element $a \in W$ with coefficients $\{\alpha_i \in A\}_i$. Then the elements $\xi_i = f(\alpha_i)$ only depend on x and not on the lift a. They are the coordinates of x. On the other hand, any sequence $\{\xi_i \in A'$ give rise to an element $x \in W'$ in a unique way.

The multiplication with p in W' is given by $(\xi_0, \xi_1, \dots) \mapsto (0, \xi_0, \xi_1, \dots)$, thus p is not a zero divisor in W'. Moreover, $\bigcap p^n W' = 0$, and therefore the p-adic topology on W' is separated. As a quotient of a complete ring, W' is also complete. Finally, the morphism, $W' \to A'$ which assignes to x its first coordinate ξ_0 descents to an automorphism $W'/p \to A'$. And this shows, that W' has residue ring A'. □

Theorem 2.10. *For every perfect ring A o characterisitic $p \neq 0$, there is a unique strict p-ring denoted by $W(A)$ with residue ring A.*

Proof. If exisctence is shown, uniqueness is Corollary 2.8.

If A is of the form $\mathbb{F}_p[X_i^{p^{-\infty}}, i \in \mathbb{N}_0]$ then $W(A) = \mathbb{Z}_p[X_i^{p^{-\infty}}, i \in \mathbb{N}_0]$. The general case follows from Lemma 2.9, if we remark that any perfect ring of characteristic p can be wriiten as a quotient of $\mathbb{F}_p[X_i^{p^{-\infty}}, i \in \mathbb{N}_0]$. Proposition 2.7 shows that this defines a functor $W(-)$ as

$$\mathrm{Hom}(A, A') \cong \mathrm{Hom}(W(A), W(A'))$$

is an isomorphism. □

Corollary 2.11. *For every perfect field k of characteristic p, there is a unique complete dvr $W(k)$, which is totally unramified and as residue field k.*

Proof. This is just a special case of Theorem 2.14 if one realises that every complete totally unramified dvr with residue field k is just a strict p-ring with residue field k. □

Corollary 2.12. *Let V be a complete dvr of mixed characteristic and perfect residue field k. Let e be the ramification index. There is a unique homomorphism $W(k) \to V$ such that the diagram*

Proof. Note that V is a (possibly non-strict) p-ring. Thus we can apply Proposition 2.7 to the identity $\mathrm{id} : k \to k$, which gives existence and uniqueness of the morphism. It is injective trivially, as V is of characteristic 0. Moreover, one can show, that if π is a local uniormiser of V, any element $y \in V$ can be written in the form

$$y = \sum_{i=0}^{-\infty} \sum_{j=0}^{e-1} [\alpha_{ij}] \pi^j p^i \quad , \quad \alpha_{ij} \in k$$

hence,$\{1, \pi, \dots, \pi^{e-1}\}$ is a basis of V as $W(k)$-module. □

Remark 2.13. Note that for the definition of addition, multiplication and subtraction on $W(A)$ via the functions Q_i^*, one has to use all $p^{n\text{th}}$ roots of the variables X_i and Y_i. Thus we had to restict ourself to perfect residue rings. To be able to generalise this, one has to define the coordinates of an element $a \in W(A)$ by the formula

$$a = \sum_{i=0}^{-\infty} [\alpha_i]^{p^{-i}} p^i.$$

This leads to the definition of Witt vectors.

2.2 The ring of p-typical Witt vectors

Let $\{X_i\}_{i\in\mathbb{N}_0}$ be a set f variables. COnsider the polynomials

$$w_n(\underline{X}) = \sum_{i=0}^{n} p^i X^{p^{n-i}}$$

called the Witt polynomials. It is clear, that one can express the X_i as polynomials in the w_n with coefficients in $\mathbb{Z}[p^{-1}]$. Let $\{Y_i\}_{i\in\mathbb{N}_0}$ be another set of variables.

Theorem 2.14. *For any polynomial $\Phi \in \mathbb{Z}[X, Z]$ there is a unique sequence of polynomials $\phi_0, \phi_1, \dots \in \mathbb{Z}[X_i, Y_j]$ such that*

$$w_n(\underline{\phi}) = \Phi(w_n(\underline{X}), w_n(\underline{Y})).$$

Proof. Existence and uniqueness are rather evident over $\mathbb{Z}[p^{-1}]$.(ϕ_n is defined recursively and uniquely by a system of n equations.) So the main task is, to show that the coefficients of the ϕ_i lie in $\mathbb{Z}$. We do this again following ideas by Lazard as explained in [12, Sec. II. 6].

Take again $\widehat{S} = \mathbb{Z}_p[\underline{X}^{p^{-\infty}}, \underline{Y}^{p^{-\infty}}]$, and set

$$x' = \sum X_i^{p^{-i}} p^i \quad \text{and} \quad y' = \sum Y_i^{p^{-i}} p^i$$

As $\Phi(x', y') \in \widehat{S}$ we can write it in a unique way in the form

$$\Phi(x', y') = \sum [\overline{\psi}_i]^{p^{-i}} p^i \quad \text{with} \quad \psi_i \in \mathbb{F}_p[\underline{X}^{p^{-\infty}}, \underline{Y}^{p^{-\infty}}]$$

Let ψ_i be representatives of $\overline{\psi}_i$ in $\widehat{S}$. One has a congruence

$$\Phi(\sum_{i\leqslant n} X_i^{p^{-i}} p^i, \sum_{i\leqslant n} Y_i^{p^{-i}} p^i) \cong \sum_{i\leqslant n} [\overline{\psi}_i]^{p^{-i}} p^i \mod p^{n+1}$$

Replacing X_i by $X_i^{p^n}$ and Y_i by $Y_i^{p^n}$, which is an automorphism of $\widehat{S}$, gives

$$\Phi(w_n(\underline{X}), w_n(\underline{Y})) \cong \sum_{i\leqslant n} [\overline{\psi}_i(\underline{X}^{p^n}, \underline{Y}^{p^n})]^{p^{-i}} p^i \mod p^{n+1}$$

But $\overline{\psi}_i(\underline{X}^{p^n}, \underline{Y}^{p^n}) = \overline{\psi}(\underline{X}, \underline{Y})^{p^n}$ as the coefficients of $\overline{\psi}$ are in $\mathbb{F}_p$. Furthermore, we know that $[-]$ commutes with p^{th} power, so

$$\Phi(w_n(\underline{X}), w_n(\underline{Y})) = w_n(\underline{\phi}) \cong \sum_{i \leqslant n} [\overline{\psi}_i]^{p^{n-i}} p^i \mod p^{n+1}$$

But $[\overline{\psi}_i] \cong \psi \mod p$ so $[\overline{\psi}_i]^{p^{n-i}} \cong \psi^{p^{n-i}} \mod p^{n-i+1}$, thus

$$w_n(\underline{\phi}) \cong w_n(\underline{\psi}) \mod p^{n+1}$$

By induction one can assume that ϕ_i for $i < n$ has integer coefficients and is congruent $\psi_i \mod p$. Then by the above congruence, one obtains

$$p^n \phi_n \cong p^n \psi_n \mod p^{n+1}$$

so that ϕ_n has integer coefficients and is congruent $\psi_n \mod p$. □

Definition 2.15. Denote now by $\underline{S} \in \mathbb{Z}[\underline{X}, \underline{Y}]$ and $\underline{P} \in \mathbb{Z}[\underline{X}, \underline{Y}]$ the polynomials associated to addition $(\Phi(X, Y) = X + Y)$ and multiplication $(\Phi(X, Y) = XY)$.

Let A by any commutative ring (with unit). By the above formulae, we define composition laws on $A^{\mathbb{N}}$ for $\underline{a} = (a_0, a_1, \ldots)$ and $\underline{b} = (b_0, b_1, \ldots)$:

$$\begin{aligned} \underline{a} + \underline{b} &= (S_0(\underline{a}, \underline{b}), S_1(\underline{a}, \underline{b}), \ldots) \\ \underline{a} \cdot \underline{b} &= (P_0(\underline{a}, \underline{b}), P_1(\underline{a}, \underline{b}), \ldots) \end{aligned}$$

Theorem 2.16. *These composition laws make $A^{\mathbb{N}}$ into a commutative ring with unit, called the ring of Witt vectors with coefficients in A, and denoted by $W(A)$.*

Proof. By definition of the $\underline{S}$ and $\underline{P}$ the Witt polynomials define a homomorphism of rings

$$\begin{aligned} w : W(A) &\to A^{\mathbb{N}} \\ (a_0, a_1, \ldots) &\mapsto (w_0(\underline{a}), w_1(\underline{a}), \ldots) \end{aligned}$$

where addition and multiplication on the right side is component wise, and on the left side by $\underline{S}$ and $\underline{P}$. It is an isomorphism, if p is invertible in A, and in this case, it is easy to see, that the unit in $W(A)$ is given by $(1, 0, 0, \ldots)$.

But if the theorem is true for a ring A, it is also true for subrings and quotients. Since it holds for $\mathbb{Z}[p^{-1}][\underline{X}]$ it is also true for $\mathbb{Z}[\underline{X}]$ and thus for any commutative ring (with unit). □

Exercise 2.17. Compute a few polynomials S_n and P_n.

We may also consider Witt vectors of inite length, by only considering the first n variables $(a_0, \ldots, a_{n-1})$, denoted by $W_n(A)$ with underlying set A^n. As the ϕ_i from the theorem only contain variables of index $\leqslant i$, this is a quotient of $W(A)$. We have $W_1(A) = A$ (rememebr this for later) and $\varprojlim W_n(A) = W(A)$.

2.3 Big Witt vectors

We will now discuss the multi-prime generalisation of Witt vectors [6]. The difference is, that we generalise the index set.

Definition 2.18. Let $S \subset \mathbb{N}$. We say that S is a truncation set, or divisor stable, if for $n \in S$, and $d \in \mathbb{N}$ a divisor of n, then $d \in S$.

Examples 2.19. $\mathbb{N}$ itself and the finite subsets $\{1, \ldots, n\}$ are truncation sets. For a prime number p, the set $\{1, p, p^2, \ldots\}$ and the finite sets $\{1, p, \ldots, p^n\}$ are truncation sets.

For a commutative ring A we define.

Definition 2.20. The big Witt ring $\mathbb{W}_S(A)$ is the set A^S equipped with the ring structure such that the ghost map defined by the Witt polynomials

$$\begin{aligned} w : \mathbb{W}_S(A) &\to A^S \\ w_n(\underline{a}) &= \sum_{d|n} d a_d^{\frac{n}{d}} \end{aligned}$$

is a natural transformation of ring functors.

As usual, on the right hand side, we take component wise addition and multiplication.

Examples 2.21. If $S = \mathbb{N}$, we write $\mathbb{W}(A) := \mathbb{W}_S(A)$. For $S = \{1 = p^0, p = p^1, p^2, \ldots\}$ for a prime number p, we obtain the ring of p-typical Witt vectors (usually indexed by the exponents of p), which we denote as usual by $W(A)$ and for a finite set $S = \{1, \ldots, n\}$ we obtain truncated Witt vectors. In particular, for $S = \{1, p, \ldots, p^n\}$, we obtain the usual ($p$-typical) truncated Witt vectors.

To prove that there exists such a ring structure, we follow a similar strategy as in the case of p-typical Witt vectors, that is, we need a criterion similar to (but more general than) Theorem 2.14 that tells us, when an element is in the image of the ghost map: roughly we have to be able to take $(p^n)^{\mathrm{th}}$ roots of representatives for all primes p.

Lemma 2.22 (Dwork)**.** *Suppose that for every prime number p, there is a ring homomorphism $\phi_p : A \to A$ such that $\phi_p(a) \equiv a^p \mod p$. Then a sequence $\{x_n \bigm| n \in S\}$ is in the image of the ghost map, if and only if $x_n \equiv \phi_p(x_{\frac{n}{p}}) \mod p^{\nu_p(n)}$ for all p, and for all $n \in S$ with $\nu_p(n) \geqslant 1$.*

Proof. It is easy (exercise!) to see that if $a \equiv b \mod p$, then $a^{p^{n-1}} \equiv b^{p^{n-1}}$ (we have already used this above). Since ϕ_p is a ring homomorphism,

$$\phi_p(w_{\frac{n}{p}}(\underline{a})) = \sum_{d|(\frac{n}{p})} d\phi_p(a_d^{\frac{n}{pd}}) \equiv \sum_{d|(\frac{n}{p})} da_d^{\frac{n}{d}} \mod p^{\nu_p(n)}.$$

The last congruence comes from the fact just stated, and because we summ over all divisors of $\frac{n}{p}$. For an integer d dividing n but not $\frac{n}{p}$, we have $\nu_p(n) = \nu_p(d)$, thus $0 \equiv d \mod p^{\nu_p(d)} \equiv d \mod p^{\nu_p(n)}$ and we can rewrite the sum $\mod p^{\nu_p(n)}$ as $\sum_{d|n} da_d^{\frac{n}{d}} = w_n(\underline{a})$. Together

$$w_n(\underline{a}) \equiv \phi_p(w_{\frac{n}{p}}(\underline{a})) \mod p^{\nu_p(n)}.$$

On the other hand, if a sequence $(x_n \mid n \in S)$ satisfies $x_n \equiv \phi_p(x_{\frac{n}{p}}) \mod p^{\nu_p(n)}$, we have to find $\underline{a}$ such that $w_n(\underline{a}) = x_n$. We do this by induction: let $a_1 = x_1$ and assume for an n all a_d with $n \neq d|n$ chosen such that $w_d(\underline{a}) = x_d$. Then

$$x_n \equiv \sum_{n \neq d|n} da_d^{\frac{n}{d}} \mod p^{\nu_p n}$$

and we can find $a_n = x_n - \sum_{n \neq d|n} da_d^{\frac{n}{d}}$. □

Proposition 2.23. *There is a unique ring structure on the set $\mathbb{W}_S(A)$ that makes the ghost map a natural transformation of ring functors.*

Proof. As done previously, we start with a polynomial ring, where the variables are indexed by S, $A = \mathbb{Z}_[X_n, Y_n \mid n \in S]$. Then the ring homomorphism given by

$$\begin{array}{rcl} \phi_p : A & \to & A \\ X_n & \mapsto & X_n^p \text{ and} \\ Y_n & \mapsto & Y_n^p \end{array}$$

satisfies the conditions of Dwork's Lemma. It follows then that for $\underline{a} \in \mathbb{W}_S(A)$ and $\underline{b} \in \mathbb{W}_S(A)$ the elements $w(\underline{a}) + w(\underline{b})$, $w(\underline{a}) \cdot w(\underline{b})$ and $-w(\underline{a})$ in $A^{\mathbb{N}}$ are in the image of the ghost map (this is clear for $\underline{a} = \underline{X}$ and $\underline{b} = \underline{Y}$ and follows then immediately as A is torsion free), so there are sequences of polynomials $(s_n^* \mid n \in S)$, $* = +, -, \cdot$, such that $w(\underline{s}^+) = w(\underline{a}) + w(\underline{b})$,etc.

For a general commutative ring A', there eis a homomorphism $f : A \to A'$ such that for $\underline{a}', \underline{b}' \in \mathbb{W}_S(A')$ the induced homomorphism

$$\mathbb{W}_S(f) : \mathbb{W}_S(A) \to \mathbb{W}_S(A')$$

sends $\underline{X} \mapsto \underline{a}$ and $\underline{Y} \mapsto \underline{b}$. Then

$$\underline{a}' * \underline{b}' = \mathbb{W}_S(f)(s^*(\underline{a}, \underline{b}))$$

and this defines the ring structure. □

Most of the additional structure from p-typical Witt vectors generalises to big Witt vectors.

The restriction map. If $T \subset S$ are both truncation sets, the forgetful functor

$$R^S_T : \mathbb{W}_S(A) \to \mathbb{W}_T(A)$$

corresponds to the restriction map. If $S = \{p^i \mid i \in \mathbb{N}_0\}$ and $T = \{p^0, \dots p^{n-1}\}$ we obtain the usual restriction map.

Verschiebung. If $n \in \mathbb{N}$ and S is a truncation set, then

$$\frac{S}{n} = \{d \in \mathbb{N} \mid nd \in S\}$$

is also a truncation set, and we define

$$\begin{aligned} V_n : \mathbb{W}_{\frac{S}{n}}(A) &\to \mathbb{W}_S(A) \\ (V_n(A_d \mid d \in \frac{S}{n}))_m &= \begin{cases} a_d & \text{if } m = nd \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

which shifts an entry a_d from the d^{th} to the $n \cdot d^{\text{th}}$ slot. For $S = \{p^0, \dots, p^n\}$, $\frac{S}{p} = \{p^0, \dots, p^{n-1}$ and

$$V_p : W_n(A) \to W_{n+1}(A)$$

is the usual Verschiebung. It is an easy (exercise!) lemma to show the V_n is additive (hint: apply the ghost map).

Frobenius. Recall that in the p-typical case, the Frobenius map could be constructed recursively, by solving polynomial equations, to make a certain diagram commute. Frobenius should make the diagram

$$\begin{tikzcd} \mathbb{W}_S(A) \arrow[r, "\mathrm{F}_n"] \arrow[d] & \mathbb{W}_{\frac{S}{n}}(A) \arrow[d] \\ A^S \arrow[r, "\mathrm{F}^w_n"] & A^{\frac{S}{n}} \end{tikzcd}$$

with $(\mathrm{F}^w_n(x_m \mid m \in S))_d = x_{nd}$ commute. First for $A = \mathbb{Z}[X_m \mid m \in S]$. Then by Dwork's Lemma with the map $\phi_p(X_i) = X_i^p$, $\mathrm{F}^w_n(w(\underline{X}))$ is again in the image of the ghost map, given by a set of polynomials $(f_i | i \in S)$, which can be determined recursively. Now we pass to a general commutative ring A' as in the proof of the ring operations.
Exercise: show that if A is an $\mathbb{F}_p$-algebra, and $\varphi : A \to A$ the Frobenius endomorphism, then the Frobenius for p on $\mathbb{W}_S(A)$ is given by the formula

$$\mathrm{F}_p = R^S_{\frac{S}{p}} \circ \mathbb{W}_S(\varphi).$$

Teichmüller representatives. The map

$$\begin{aligned} [-]_S : A &\to \mathbb{W}_S(A) \\ ([a]_S)_n &= \begin{cases} a & \text{if } n = 1 \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

is multiplicative, making the diagram

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, "{[-]_S}"'] & A \arrow[d, "{[-]^w_S}"'] \\ \mathbb{W}_S(A) \arrow[r, "w"] & A^S \end{tikzcd}$$

with $([a]^w_S)_n = a^n$ commutative.

Relations. The following relations are easy to verify (exercise!). Let $\underline{a}, \underline{a}' \in \mathbb{W}_S(A)$.

$$\begin{aligned} \underline{a} &= \sum_{n\in S} V_n([a_n]_{\frac{S}{n}}) \\ \mathrm{F}_n V_n(\underline{a}) &= n\underline{a} \\ \underline{a}V_n(\underline{a}') &= V_n(\mathrm{F}_n(\underline{a})\underline{a}') \\ \mathrm{F}_m V_n &= V_n \mathrm{F}_m \quad \text{ if } (m,n)=1 \end{aligned}$$

Exercise: show that

$$\mathbb{W}_S(\mathbb{Z}) = \prod_{n\in S} \mathbb{Z}\cdot V_n([1]_{\frac{S}{n}}).$$

Projective limit. Let S be a truncation set. Then by definition

$$\mathbb{W}_S(A) = \lim_{T\subset S \text{ finite}} \mathbb{W}_T(A).$$

Decomposition. Let p be a prime and denote by $P=\{1,p,p^2,\dots\}$. Let $I(S)=\{k\in S \mid p\nmid k\}$. Assume further, that every $k\in I(S)$ is invertible in A. Then there is a natural idempotent decomposition

$$\mathbb{W}_S(A) = \prod_{k\in I(S)} \mathbb{W}_{\frac{S}{k}\cap P}(A).$$

Functoriality. Let again $A=\mathbb{Z}[X_n \mid n\in S]$ then for any ring B there is a natural identification

$$\mathrm{Hom}(A,B)\cong \mathbb{W}_S(B)$$

meaning that $\mathbb{W}_S(-)$ is representable. The ring structure on $\mathbb{W}_S(B)$ makes R into a ring object in the category of $\mathbb{Z}$-algebras.

Remark 2.24. Witt–Burnside rings are a generalisation of Witt vectors using pro finite groups G. In this set-up the usual p-typical Witt vectors correspond to $G=\mathbb{Z}_p$. Examples for $G=\mathbb{Z}_p^n$ can be thought of as tree version of $W(-)$. Examples are extremely hard to compute, and not many applications are known.

Remark 2.25. Consider the natural projection

$$\begin{aligned} \epsilon : \mathbb{W}(A) &\to A \\ \underline{a} &\mapsto a_1 \end{aligned}$$

There is a unique natural ring homomorphism

$$\Lambda : \mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(A))$$

such that $w_n(\Lambda(a)) = \mathrm{F}_n(a)$ for all $n\in\mathbb{N}$.

The element $(\mathrm{F}_n(a))_{n\in\mathbb{N}} \in \mathbb{W}(A)^{\mathbb{N}}$ is in the image of the ghost map according to Dworks Lemma (use that $\mathrm{F}_p : \mathbb{W}(A \to \mathbb{W}(A)$ satisfies $\mathrm{F}_p(a)\equiv a^p \mod p\,\mathbb{W}(A))$. This determines the map Λ such that

$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "\Lambda"] \arrow[dr, "(\mathrm{F}_n)_n"'] & \mathbb{W}(\mathbb{W}(A)) \arrow[d, "w"] \\ & \mathbb{W}(A)^{\mathbb{N}} \end{tikzcd}$$

Moreover, the triple $(\mathbb{W}(-),\Lambda,\epsilon)$ form a comonad on the category of rings. This means that

$$\begin{aligned} \mathbb{W}(\Lambda_A)\circ\Lambda_A &= \Lambda_{\mathbb{W}(A)}\circ\Lambda_A : \mathbb{W}(A)\to\mathbb{W}(\mathbb{W}(\mathbb{W}(A))) \\ \mathbb{W}(\epsilon_A)\circ\Lambda_A &= \epsilon_{\mathbb{W}(A)}\circ\Lambda_A : \mathbb{W}(A)\to\mathbb{W}(A) \end{aligned}$$

(A monad is in some sense a monoid object in a bicategory, a command is a monad in the dual category.) A special λ-ring is a ring A together with a map $\lambda : A\to\mathbb{W}(A)$ that makes A into a coalgebra over the comonad $(\mathbb{W}(-),\Lambda,\epsilon)$. For such a ring we can then define the n^{th} Adams operation by $\psi_n = w_n\circ\lambda : A\to \mathbb{W}(A)\to A$.

3 Crystalline cohomology

As we have mentioned, one of the objectives to construct a de Rham–Witt complex was to be able to compute crystalline cohomology more explicitly. In this section, we want to give a quick review of the basic concepts of crystalline cohomology. The standard reference for crystalline cohomology is of course Pierre Berthelot and Arthur Ogus' book [2]. A ver quick and to the point overview can be found in Antoine Chambert-Loir's survey article [5] and in Luc Illusie's paper [9].

3.1 Divided powers

The idea of crystalline cohomology goes back, as so many concepts in algebraic geometry, to Grothendieck. It was clear, at a very early stage of the idea, that so called divided powers would be needed for the construction, as it basically concerns an integration process.

Definition 3.1. Let A be a ring and $I \subset A$ an ideal. A PD-structure on I is a sequence of maps $\gamma_n : I \to A$ such that
— $\gamma_0(x) = 1$ and $\gamma_1(x) = x$ for all $x \in I$
— $\gamma_n(x) \in I$ for $n \geqslant 1$ and $x \in I$
— $\gamma_n(x+y) = \sum_{i+j=n} \gamma_i(x)\gamma_j(y)$ for all $x, y \in I$
— $\gamma_n(\lambda x) = \lambda^n \gamma_n(x)$ for all $\lambda \in A$ and $x \in I$
— $\gamma_n(x)\gamma_m(x) = \binom{m+n}{n}\gamma_{m+n}(x)$ for all $x \in I$ and $m, n \in \mathbb{N}$
— $\gamma_m(\gamma_n(x)) = \frac{(mn)!}{m!(n!)^m}\gamma_{mn}(x)$ for all $x \in I$ and $m, n \in \mathbb{N}$
In this case, we say that A is a PD-ring.

Where do these formulae come from? They ensure that morally "$\gamma_n(x) = \frac{x^n}{n!}$". These elements are needed to integrate — which should be clear if we just recall basic formulae from Calculus.

Examples 3.2.
1. For a perfect ring A of characteristic $p > 0$, the ideal (p) in the ring of Witt vectors $W(A)$ has a natural PD-structure, given by $\gamma_n(p) = \frac{p^n}{n!}$ which makes sense, sind the p-adic valuation of $\frac{p^n}{n!}$ is positive for all $n \in \mathbb{N}_0$ and strictly positive for $n \geqslant 1$.
2. For any ring A, we define an A-PD-algebra in n variables
$$A\langle x_1, \dots x_n\rangle = \bigoplus_{r \geqslant 0} \Gamma^r$$
where a base of Γ^r as A-modules is given by symbols $x_1^{[k_1]} \cdots x_n^{[k_n]}$ such that $k_1 + \dots k_n = r$, $k_i \in \mathbb{N}_0$. The algebra structure is given by the relations $x_i^{[m]} x_i^{[n]} = \binom{m+n}{n} x_i^{[m+n]}$. The ideal $I = A^+\langle x_1, \dots, x_n\rangle = \bigoplus_{r \geqslant 1} \Gamma^r$ then has a unique PD-structure such that $\gamma_r(x_i) = x_i^{[r]}$.

Remark 3.3. Note that if A is annihilated by a $n \geqslant 2$, then a PD ideal $I \subset A$ is automatically a nil-ideal, since $x^n = n!\gamma_n(x) = 0$ for every $x \in I$. In particular $\operatorname{Spec} A$ and $\operatorname{Spec} A/I$ have the same underlying topological space.

The idea behind crystalline cohomology is to locally compute de Rham-type complexes with additional PD-structure. Let's take the non-PD setting as a model:
Let $\mathscr{T}$ be a topos and A a (commutative unital) ring of $\mathscr{T}$.

Definition 3.4. We call an anticommutative graded A-algebra B, in positive degrees, with an A-linear differential $d : B^i \to B^{i+1}$ such that $d^2 = 0$ and $d(xy) = (dx)y + (-1)^i x dy$, a differential graded A-algebra B. A morphism of differential graded A-algebras is a morphism of A-algebras compatible with the differential structures.

Recall that for an A-algebra R the de Rham complex $\Omega_{R/A}$ is universal in the sense that for any A-dga B, every A-algebra morphism $R \to B^0$ extends in a unique way to an A-dga morphism $\Omega_{R/A} \to B$.

Proposition 3.5. *Let A be as above and denote by $\mathcal{dga}^{\geqslant 0}(A)$ the category of differential graded A algebras. The functor*
$$\mathcal{Alg}(A) \to \mathcal{dga}^{\geqslant 0}(A)\,,\ C \mapsto \Omega_{C/A}$$
is left adjoint to the forgetful functor
$$\mathcal{dga}^{\geqslant 0}(A) \to \mathcal{Alg}(A)\,,\ B \to B^0.$$

We also say, the object $\Omega_{C/A}$ is initial in the category $\mathit{dga}^{\geqslant 0}(A)$.

Definition 3.6. Let B be an A-dga. A differential graded B-module (or B-dgm) is a graded B-module M together with a differential $d : M^i \to M^{i+1}$ such that $d^2 = 0$ and $d(bx) = (db)x + (-1)^i bdx$ for $b \in B^i$ and $x \in M^j$. A morphism of B-dgm's is a morphism of B-modules compatible with the differential structure. We can define left and right B-dga's. Every right B-dgm can be seen as a left B-dgm via the anti-commutative law $bx = (-1)^{ij}xb$.A differential graded ideal (dgi) of B is a sub B-dgm of B.

If $I^0 \subset B^0$ is an ideal, then the ideal in B generated by I^0 and dI^0 is a dgi of B with zero component I^0, and it's the smallest dgi with this property (it is in fact the dgi generated by I^0). Furthermore, for $n \in \mathbb{N}$, I^n is generated additively by elements of the form $bdx_1 \cdots dx_n$ with $b \in B^0$ and $x_i \in I^0$. If I is a B-dgi, B/I is an A-dga.

Definition 3.7. Let E be a B^0-module. A connection on E with respect to B is a morphism

$$\nabla : E \to E \otimes_{B^0} B^1$$

such that $\nabla(bx) = b\nabla x + x \otimes db$.

Every connection ∇ extends in a unique way to a morphism $\nabla : E \otimes_{B^0} B^i \to E \otimes_{B^0} B^{i+1}$ such that $\nabla(b \otimes x) = b\nabla x + x \otimes db$ for $b \in B^i$ and $x \in E$.

Definition 3.8. We say that ∇ is integrable if $\nabla^2 = 0$. If this is the case, $(E \otimes B, \nabla)$ is a B-dgm

We want to take this idea to the PD-world.

Definition 3.9. Let (B, I, γ) ba an A-PD-algebra. The ideal of $\Omega_{B/A}$ generated by the elements $d(\gamma_n(x)) - \gamma_{n-1}(x)dx$ for $x \in I$ is a dgi J. Thus the quotient

$$\Omega_{B/A,\gamma} := \Omega_{B/A}/J$$

is an A-dga called the PD-de Rham complex of B/A.

It is the initial object in the category of PD-A-dga's: if C is an A-dga with a PD-ideal K of C^0 and PD-structure δ compatible with d in the sense that $d(\delta_n x) = \delta_{n-1}(x)dx$, then any morphism of A-PD-algebras $f^0 : B \to C^0$ extends uniquely to a homomorphism of A-dga's $f : \Omega_{B/A,\gamma} \to C$. Now let (A, I, γ) be a PD-ring in $\mathscr{T}$, B an A-algebra, $J \subset B$ an ideal. Let $\overline{B} = D_{B,\gamma}(J)$ be the decided power envelope of (B, J) with respect to γ (this is $B\langle J\rangle$ from the example above modes out by relations, that make the PD-structure compatible with γ). Denote by $\overline{J}$ the the associated PD-ideal. $\overline{B}$ is generated as B-algebra by the divided powers $x^{[n]}$, for $x \in J$.

Proposition 3.10. *The derivation $d : B \to \Omega^1_{B/A}$ extends in a unique way to a derivation $d : \overline{B} \to \overline{B}\Omega^1_{B/A}$ such that*

$$dx^{[n]} = x^{[n-^1} \otimes dx,$$

for $x \in J$ and $n \in \mathbb{N}$.

In [2] this comes out of the theory of hyper PD-stratifications, but it can also be verified directly.

The derivation $d : \overline{B} \to \overline{B} \otimes_B \Omega^1_{B/A}$ then extends uniquely to $\overline{B} \otimes_B \Omega_{B/A}$ and $d^2 = 0$. The universality of the A-dga $\Omega_{\overline{B},A,[-]}$ shows that there is a unique homomorphism

$$\Omega_{\overline{B},A,[-]} \to \overline{B} \otimes_B \Omega_{B/A} \tag{3.1}$$

which is the identity in degree zero.

Proposition 3.11. *The homomorphism (3.1) is an isomorphism.*

Proof. The homomorphism of grade A-aglebras

$$\overline{B} \otimes_B \Omega_{B/A} \to \Omega_{\overline{B}/A,[-]}$$

which is the identity in degree zero and given by the composition

$$\overline{B} \otimes_B \Omega^1_{B/A} \to \Omega^1_{\overline{B}/A} \to \Omega^1_{\overline{B}/A,[-]}$$

is compatible with the differential and therefore an inverse of the morphism in question. ☐

3.2 Crystalline site and crystalline cohomology

Let S be a scheme such that p is locally nilpotent, I a quasi-coherent ideal of $\mathscr{O}_S$, and γ a PD-structure on I — in other words (S,I,γ) is a PD-scheme. Think of $S=W_n(S_0)$ for S_0 the Spec of a perfect field. Let X be an S-scheme such that γ extends to a PD-structure on X. We will define the crystalline site of X with respect to (S,I,γ). The objects are S-PD-thickenings of Zariski open subsets of X.

The crystalline site of X over S is denoted by $\mathrm{Cris}(X/S)$.

— The objects are triples (U,T,δ), where U is a Zariski open of X, T is an S scheme together with a closed immersion $U\hookrightarrow T$ given by an ideal J with PD-structure δ compatible with γ (thus J is a nil-ideal and U and T have the same underlying topological space.

— The morphisms are morphisms of triple $(U,T,\delta)\to(U',T',\delta')$ sending $U\to U'$ and $T\to T'$ compatible with the PD-structure.

— The covering families are $(U_\alpha,T_\alpha,\delta_\alpha)\to(U,T,\delta)$ such that the T_α cover T.

The associated tops is denoted by $(X/S)_{\mathrm{cris}}$. One can describe a sheaf $\mathscr{E}$ on the crystalline site explicitly, by giving for each (U,T,δ) a sheaf $\mathscr{E}_{(U,T,\delta)}$ on T for the Zariski topology, and for each map $f:(U',T',\delta')\to(U,T,\delta)$ a transition map $f^*\mathscr{E}_{(U,T,\delta)}\to\mathscr{E}_{(U',T',\delta')}$ which satisfies transitivity and is an isomorphism if $T'\to T$ is an open immersion.

Examples 3.12. The structure sheaf $\mathscr{O}_{X/S}$ is given by the cofunctor $(U,T,\delta)\mapsto\mathscr{O}_T$. But also the cofunctor $(U,T,\delta)\mapsto\mathscr{O}_U$ defines a sheaf of rings denoted by $\mathscr{O}_X$. And the PD-ideal sheaf $\mathscr{J}_{X/S}\subset\mathscr{O}_{X/S}$ that associated to (U,T,δ) the defining ideal of the closed immersion $U\hookrightarrow T$, $(U,T,\delta)\mapsto\mathrm{Ker}(\mathscr{O}_T\to\mathscr{O}_U)$. In fact, there is a short exact sequence

$$0\to\mathscr{J}_{X/S}\to\mathscr{O}_{X/S}\to\mathscr{O}_X\to 0.$$

Definition 3.13. A sheaf of $\mathscr{O}_{X/S}$-modules is a crystal if all the transition morphisms are isomorphisms.

It is preferable to work with the crystalline topos as opposed to the crystalline site, because one has more functoriality: one has for example inverse image sheaves. But this needs some checking and abstract nonsense.

Example 3.14. An example to keep in mind is that of a scheme X over a perfect field K of characteristic $p>0$, and $S=W_n(k)$ with the canonical PD-structure. Then the objects of $\mathrm{Cris}(X/W_n)$ are given by diagrams

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d] & T \arrow[d] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} W_n \end{tikzcd}$$

such that the ideal $\mathrm{Ker}(\mathscr{O}_T\to\mathscr{O}_U)$ has a PD-structure compatible with the canonical Witt vector PD-structure.

To define the global section functor recall that for a topos $\mathscr{T}$ and $T\in\mathscr{T}$, $\Gamma(T,-)$ is the functor $F\mapsto\mathrm{Hom}_{\mathscr{T}}(F,T)$. If e is the final object in $\mathscr{T}$, we write $\Gamma(e,F)=:\Gamma(\mathscr{T},F)=:\Gamma(F)$. The final object for a topos is the sheafification of the constant pre sheaf given by $\{0\}$ on each U. For an ordinary topological space X this sheaf is represented by the open subset X of X itself. In case of the crystalline topos, it is not representable however. In general, a section $s\in\Gamma(\mathscr{T},F)=\mathrm{Hom}(e,F)$ is a compatible collection of sections $s_T\in F(T)$ for every $T\in X$, i.e. an element in $\varprojlim_{T\in X}F(T)$.

Let X_{Zar} be the Zariski topos of X. Then there is a canonical projection

$$u_{X/S}:(X/S)_{\mathrm{cris}}\to X_{\mathrm{Zar}}$$

given by

$$\begin{aligned} u_{X/S*}: &\qquad \Gamma(U,u_{X/S,*}\mathscr{E})=\Gamma((U/S)_{\mathrm{cris}},\mathscr{E}) \\ u_{X/S}^{-1}: &\qquad (u_{X/S}^{-1}(\mathscr{F}))_{(U,T,\delta)}=\mathscr{F}\big|_U \end{aligned}$$

It is clear, that $u_{X/S}^{-1}$ commutes with arbitrary inverse limits, so that we really have a morphism of topoi, but not of ringed topoi. It is a morphism of ringed topoi if X is considered with the sheaf $f^{-1}\mathscr{O}_S$ (for

$f : X \to S$). If $f_{\mathrm{cris}} : (X/S)_{\mathrm{cris}}? \to S$ is the projection, then there is a canonical isomorphism in the derived category

$$Rf_{\mathrm{cris}}\,\mathscr{E} = Rf_* Ru_{X/S*}\,\mathscr{E}$$

In particular, $R\Gamma(X_{\mathrm{Zar}}, Ru_*\,\mathscr{E}) \cong R\Gamma((X/S)_{\mathrm{cris}}, \mathscr{E})$.

Recall now the calculus of $(X/S)_{\mathrm{cris}}$ in case there is a closed immersion $j : X \to Z$ into a smooth scheme. In general the ideal $\mathrm{Ker}(\mathscr{O}_Z \to \mathscr{O}_X)$ does not have divided powers,thus we consider the PD-envelope$\overline{Z}$ of X in Z, meaning, that we formally add divided powers to the defining ideal in a universal way, and obtain $X \hookrightarrow \overline{Z} \to Z$. Moreover for a crystal $\mathscr{E}$ there is a unique integrable connection

$$d : \mathscr{E}_{\overline{Z}} \to \mathscr{E}_{\overline{Z}} \otimes \Omega^1_{Z/S}$$

compatible with the PD-structure. If $\mathscr{E} = \mathscr{O}_{X/S}$ this gives just the complex $\mathscr{O}_{\overline{Z}} \otimes \Omega_{Z/S} = \Omega_{\overline{Z}/S,[-]}$. A fundamental theorem of Berthelot and Grothendieck says:

Theorem 3.15. *There is a canonical isomorphism*

$$Ru_{X/S*}\,\mathscr{E} \xrightarrow{\sim} \mathscr{E}_{\overline{Z}} \otimes \Omega_{Z/S}.$$

In particular, for $\mathscr{E} = \mathscr{O}_{X/S}$ this isomorphism is compatible with the natural product structures on both sides. The proof uses a simplicial complex called the Čech-Alexander complex and the so-called crystalline Poincaré lemma. Even if globally X is not smoothable, it is locally, and using cohomological descent, we can treat this case as well.

Lemma 3.16. *Let A be a ring. The de Rham complex of $A[t_1, \dots, t_n]$ with coefficients in $A\langle t_1, \dots, t_n\rangle$ (with the integrable connection $t_i^{[k]} \mapsto t_i^{[k-1]} dt_i$) is a resolution of A.*

Now let $S = W_n$. If X has a smooth lift over W_n, crystalline cohomology of X corresponds to the de Rham cohomology of the oft.

Corollary 3.17. *If Z/W_n is a smooth lift of X, then $\overline{Z} = Z$ and*

$$H^*_{cris}(X/Wn) = H^*_{dR}(Z/W_n).$$

The isomorphism of Theorem **??** is functorial in X and compatible with base change of (S, I, γ). In particular, let X/k and $S = W_n$ with Frobenius σ. Then the absolute Frobenius of X, $\mathrm{F} : X \to X$ induces a σ-linear morphism in cohomology

$$\mathrm{F} : H^*(X/W_n) \to H^*(X/W_n).$$

4 The p-typical de Rham–Witt complex

Most of what we say here is taken from Illusie's paper [9]. If X is a smooth $\mathbb{F}_p$-scheme, one could naively try to take the de RHam complex of $W(X)$, and compute the hypercohomology. But it turns out that this doesn't work — it is not even compatible with taking the limit $\lim W_n(\mathscr{O}_X) = W(\mathscr{O}_X)$ (it is not functorial in X). On the other hand the limit of the de Rham complexes of $W_n(X)$ is not compatible with Frobenius and Verschiebung. Thus Deligne's idea was to extend the projective system $W_\bullet(\mathscr{O}_X)$ to a projective system of dga's $W_\bullet\Omega_X$) and also extend the operators F and V satisfying suitable equalities.

4.1 Definition for $\mathbb{F}_p$-algebras

Following the intuition from the de Rham complex, we will define the de Rham-WItt complex as initial object in a certain category.

Definition 4.1. Let X be a topos. A de Rham-V-procomplex is a projective system

$$M_\bullet = ((M_n)_{n\in\mathbb{Z}}, R : M_{n+1} \to M_n)$$

of $\mathbb{Z}$-dga's on X and a family of additive maps

$$(V : M_n^i \to M_{n+1}^i)_{n\in\mathbb{Z}}$$

such that $RV = VR$ satisfying the following conditions:

(V1) $M_{n\leqslant 0}=0$, M_1^0 is an $\mathbb{F}_p$-algebra and $M_n^0=W_n(M_1^0)$ where R and V are the usual maps.

(V2) For $x\in M_n^i$ and $y\in M_n^j$

$$V(xdy)=(Vx)dVy.$$

(V3) For $x\in M_1^0$ and $y\in M_n^0$

$$(Vy)d[x]=V([x]^{p-1}y)dV[x].$$

A morphism of de Rham-V-procomplexes is a morphism of a projective system of dga's $(f_n:M_n\to M_n')_n$ compatible with all the additional structure in the obvious way ($f_{n+1}V=Vf_n$ and $f_n^0=W_n(f_1^0)$). Thus the de Rham-V-procomplexes form in a natural way a category denoted by $\mathrm{VDR}(X)$. there is a forgetful functor

$$\mathrm{VDR}(X)\to\mathbb{F}_p\,\mathcal{Alg}(X)\quad,\quad M_\bullet\mapsto M_1^0 \tag{4.1}$$

We can now explain the construction of the de Rham–Witt complex.

Theorem 4.2. *The forgetful functor (4.1) has a left adjoint $A\mapsto W_\bullet\Omega_A$: there is a functorial isomorphism*

$$\mathrm{Hom}_{\mathrm{VDR}(X)}(W_\bullet\Omega_A,M_\bullet)\cong\mathrm{Hom}_{\mathbb{F}_p\,\mathcal{Alg}(X)}(A,M_1^0).$$

For $n\in\mathbb{N}$ the morphism of $\mathbb{Z}$-dga's $\pi_n:\Omega_{W_n(A)}\to W_n\Omega_A$ such that $\pi_n^0=\mathrm{id}$ is surjective and $\pi:\Omega_A\to W_1\Omega_A$ is an isomorphism.

Proof. The construction is inductive in n. Let $W_n\Omega_A=0$ for $n\leqslant 0$. Then set $W_1\Omega_A=\Omega_A$. Assume that for fixed $n\geqslant 0$ the system $(R:W_i\Omega_A\to W_{i-1}\Omega_A)_{i\leqslant n}$ and the maps $(V:W_{i-1}\Omega_A\to W_i\Omega_A)_{i\leqslant n}$ are constructed, such that the following conditions are satisfied

$(0)_n$ $RVx=VRx$ for $x\in W_i\Omega_A$, $i\leqslant n-1$.

$(1)_n$ $W_i\Omega_A^0=W_i(A)$ for $i\leqslant n$ and there V and R are as usual.

$(2)_n$ $V(xdy)=(Vx)dVy$ for $x,y\in W_i\Omega_A$, $i\leqslant n-1$.

$(3)_n$ $(Vy)d[x]=V([x]^{p-1}y)dV[x]$ for $x\in A$, $y\in W_i(A)$, $i\leqslant n-1$.

$(4)_n$ $\pi\Omega_{W_i(A)}\to W_i\Omega_A$ is an epimorphism for $i\leqslant n$.

Now we construct $W_{n+1}\Omega_A$ together with R and V satisfying $(0)_{n+1},\dots,(4)_{n+1}$.

Let $v:W_n(A)^{\otimes i+1}\to\Omega_{W_{n+1}(A)}$ given by

$$(a\otimes x_1\otimes\cdots\otimes x_i)\mapsto VadVx_1\dots dVx_i$$

and $\varepsilon:W_n(A)^{\otimes i+1}\to\Omega^i_{W_n(A)}$ by

$$(a\otimes x_1\otimes\cdots\otimes x_i)\mapsto adx_1\dots dx_i$$

Let K^i be the kernel of the composition

$$W_n(A)^{\otimes i+1}\xrightarrow{\varepsilon}\Omega^i_{W_n(A)}\xrightarrow{\pi_n}$$

then $\oplus_i v(K^i)$ is a graded ideal of $\Omega_{W_n(A)}$ (but not stable by d in general). Furthermore, let I be the $W_{n+1}(A)$-submodule of $\Omega^1_{W_{n+1}(A)}$ generated by sections of the form $Vy.d[x]-V([x]^{p-1}y)dV[x]$. Let N be the dgi of $\Omega_{W_{n+1}(A)}$ generated by I and $\oplus_i v(K^i)$. Then we define

$$W_{n+1}\Omega_A:=\Omega_{W_{n+1}(A)}/N$$

and π_{n+1} is then just the projection $\Omega_{W_{n+1}(A)}\to W_{n+1}\Omega_A$. The restriction $R:W_{n+1}(A)\to W_n(A)$ induces a morphism of dga's

$$R:\Omega_{W_{n+1}(A)}\to\Omega_{W_n(A)}$$

and because $\pi_nR(N)=0$ it induces a morphism on the quotients

$$RW_{n+1}\Omega_A\to W_n\Omega_A.$$

Moreover, since by construction $\pi_{n+1}v(K^i)=0$, V induces an additive map

$$V:W_n\Omega_A\to W_{n+1}\Omega_A$$

satisfying the desired properties. The remaining properties $(0)_{n+1}, \ldots, (4)_{n+1}$ are easily verified.

It remains to show that the constructed complex satisfies the desired universal property.

Let $M_\bullet$ be a de Rham-V-procomplex and $f_1^0 : A \to M_1^0$ a homomorphism. Then there is a unique $f_1 : \Omega_A \to M_1$ of dga's extending f_1^0. Inductively, we construct $f_\bullet$.

Assume for $n \geqslant 1$ the morphisms of dga's $f_i : W_i\Omega_A \to M_i$ for $i \leqslant n$ constructed (uniquely because π_i is surjective) such that $f_{i-1}R = Rf_i$, $Vf_{i-1} = f_iV$ and $f_i^0 = W_i(f_1^0)$.

Let $g_{n+1} : \Omega_{W_{n+1}(A)} \to M_{n+1}$ the unique morphism of dga's that extends $W_{n+1}(f_1^0) = f_{n+1}^0$. Then $g_{n+1}(N) = 0$ and the induced map on the quotient $f_{n+1} : W_{n+1}\Omega_A \to M_{n+1}$ satisfies $f_nR = Rf_{n+1}$ and $Vf_n = f_{n+1}V$. The resulting family $f_\bullet$ extends f_1^0 uniquely to a morphism of $\mathrm{VDR}(X)$. □

Definition 4.3. Let A be an $\mathbb{F}_p$-algebra of X. The de Rham-V-procomplex $W_\bullet\Omega_A$ is called the de Rham–Witt pro complex of A.

4.2 Some properties

Proposition 4.4. *Let A be as above.*

$$\begin{aligned} xVy &= V(\mathrm{F}\,Rx.y) \quad \text{for } x \in W_n(A), y \in W_{n-1}\Omega_A^i \\ (d[x])Vy &= V(([x]^{p-1}d[x])y) \quad \text{for } x \in A, y \in W_{n-1}\Omega_A^i \end{aligned}$$

Proof. This follows because of the surjectivity directly from (V3) and (V2). □

Proposition 4.5. *Let A be a perfect $\mathbb{F}_p$-algebra. Then $W_\bullet\Omega_A^i = 0$ for $i > 0$.*

Proof. Because of the subjectivity of π it suffices to show this for $\Omega^i_{W_n(A)}$ for $i > 0$ and every n. In fact for a $W_n(A)$-module M any derivation $d : W_n(A) \to M$ is zero: Let $\underline{x} = (x_0, \ldots x_{n-1}) \in W_n(A)$. This can be written as the sum $\underline{x} = [x_0] + V[x_1] + \ldots + V^{n-1}[x_{n-1}]$, and thus

$$\mathrm{F}^n\,\underline{x} = [x_0]^{p^n} + p[x_1]^{p^{n-1}} + \ldots + p^{n-1}[x_{n-1}]^p$$

and $d\,\mathrm{F}^n\,\underline{x}$ is divisible by p^n, and therefore zero. But by hypothesis F is an automorphism (of A), and it follows that d is already zero. □

By construction $W_\bullet\Omega(A)$ is functorial in A, and any morphism of $\mathbb{F}_p$-algebras on X $u : A \to B$ induces a morphism in $\mathrm{VDR}(X)$

$$W_\bullet\Omega_u : W_\bullet\Omega_A \to W_\bullet\Omega_B$$

In particular if k is perfect of characteristic p and A a k-algebra, then $W_n\Omega_A$ is naturally a $W_n(k)$-dga (i.e. d is $W_n(k)$-linear), and V is $\sigma^{-1}W_\bullet(k)$-linear.

Let $k \to k'$ be a morphism of perfect rings of characteristic p and A a k-algebra and $A' = A \otimes k'$, then there is a morphism

$$W_\bullet\Omega_A \otimes W_\bullet(k') \to W_\bullet\Omega_{A'}.$$

Proposition 4.6. *This morphism is an isomorphism.*

Proof. Show this first for the Witt vectors. For this we need that the square

$$\begin{array}{ccc} A' & \xrightarrow{\mathrm{F}} & A' \\ \uparrow & & \uparrow \\ A & \xrightarrow{\mathrm{F}} & A \end{array}$$

is cocartesian, which it is, because k' is perfect. Because we have isomorphisms of dga's

$$\oplus_{n\in\mathbb{N}_0} F_*^n A \xrightarrow{\sim} \mathrm{gr}_V\, W(A)$$

and similar for A', it follows that for each $n \in \mathbb{N}$

$$W_n(A) \otimes_{W_n(k)} W_n(k') \cong W_n(A')$$

.

Then show that the left hand side is a de Rham-V-procomplex (for this we have to define a Verschiebung:

$$V : W_n\Omega^i_A \otimes W_n(k') \to W_{n+1}\Omega^i_A \otimes W_{n+1}(k') \quad , \quad V(x \otimes FRy) = Vx \otimes y$$

which is the usual V in degree 0). and use universality to extend the identity on A' uniquely to a morphism

$$W_\bullet \Omega_{A'} \to W_\bullet \Omega_A \otimes W_\bullet (k')$$

which is the inverse of the canonical morphism above. □

The functor $W_n(-)$ commutes with inductive filtering limits of $\mathbb{F}_p$-algebras on X. It follows that the category VDR(X) has filtering inductive limits and if $(A_i)_i$ a filtering inductive system with $A = \varinjlim A_i$, the canonical map

$$\varinjlim W_\bullet \Omega_{A_i} \to W_\bullet \Omega_A$$

is an isomorphism.

In particular, if U is an object of X, the $\Gamma(U, W_\bullet \Omega_A)$ is a de Rham-V-procompelx and

$$W_\bullet \Omega_{\Gamma(U,A)} \to \Gamma(U, W_\bullet \Omega_A)$$

extends the identity in degree zero. This defines a morphism of presheaves which induces an isomorphism on the associated sheaves.

Similar to a statement above, but important in the light of sheaf theory:

Proposition 4.7. *Let $A \to B$ a localisation morphism of $\mathbb{F}_p$-algebras on X (identify B with $S^{-1}A$). Then the $W_\bullet(B)$-linear map*

$$W_\bullet (B) \otimes W_\bullet \Omega^i_A \to W_\bullet \Omega^i_B$$

is an isomorphism

Proof. The idea is similar to above: to show it in degree 0, we need again that the square

$$\begin{tikzcd} B \arrow[r, "\mathrm{F}"] & B \\ A \arrow[u] \arrow[r, "\mathrm{F}"] & A \arrow[u] \end{tikzcd}$$

is cocartesian (which it is, because we are dealing with a localisation morphism, and $(S^p)^{-1}A = S^{-1}A = B$). Then show that the left hand side is a de Rham-V-procomplex in order to use universality to get an inverse to the morphism in question. □

Now let $(X, \mathscr{O}_X)$ be a ringed tops of $\mathbb{F}_p$-algebras. Then the de Rham–Witt procomplex of $\mathscr{O}_X$ is denoted by

$$W_\bullet \Omega_X.$$

If $f : X \to Y$ is a morphism of ringed topoi of $\mathbb{F}_p$-algebras, then $f_* W_\bullet \Omega_X$ and $f^{-1} W_\bullet \Omega_Y$ are naturally de Rham-V-procomplexes, and there are adjoint maps

$$\begin{array}{rcl} W_\bullet \Omega_Y & \to & f_* W_\bullet \Omega_X \\ f^{-1} W_\bullet \Omega_Y & \to & W_\bullet \Omega_X \end{array}$$

If $\mathscr{O}_X = f^{-1}\mathscr{O}_Y$, the second one is an isomorphism. And in particular, for a point $x \in X$

$$(W_\bullet \Omega_X)_x \to W_\bullet \Omega_{X,x}$$

Proposition 4.8. *For each $n \in \mathbb{N}$ $W_n\Omega^i_X$ is a quasi-coherent sheaf of $W_n(X)$. For each open affine, $U = \operatorname{Spec} A$, we have $\Gamma(U, W_n\Omega^i_X) = W_n\Omega^i_A$.*

Proof. Use the classical methods from basic algebraic geometry. □

Proposition 4.9. *Let $f : X \to Y$ be an étale morphism of $\mathbb{F}_p$-schemes. Then for each n, the $W_n(\mathscr{O}_X)$-linear map*

$$f^* W_n\Omega^i_Y \to W_n\Omega^i_X$$

is an isomorphism.

Proof. It is enough to show this for affine schemes. In this case we have $f : A \to B$ and have to show that

$$W_n(B) \otimes W_n\Omega^i_A \to W_n\Omega^i_B$$

is an isomorphism. For the Witt vectors, we identify again $\mathrm{gr}_V W_n(A)$ with $\oplus_{m<n} F^m_* A$ and similar for B, and we have an isomorphism $B \otimes \mathrm{gr}_V W_n(A) \cong \mathrm{gr}_V W_N(B)$. Moreover, $W_n(f)$ is étale and

$$\begin{tikzcd} W_n(B) \arrow[r, "F"] & W_n(B) \\ W_n(A) \arrow[r, "F"] \arrow[u] & W_n(A) \arrow[u] \end{tikzcd}$$

is cocartesian.

Because $W_n(B)$ is étale over $W_n(A)$, the derivation of $W_n\Omega_A$ extends uniquely to a derivation on $W_n(B) \otimes W_n\Omega_A$ by

$$d(b \otimes x) = (db)x + b \otimes dx$$

where db is the image of the composition

$$W_n(B) \xrightarrow{d} \Omega^1_{W_n(B)} = W_n(B) \otimes \Omega^1_{W_n(A)} \to W_n(B) \otimes W_n\Omega^1_A.$$

Thus we obtain a projective system of dga's $W_\bullet(B) \otimes W_\bullet\Omega_A$.

To obtain the Verschiebung operator, because the above diagram is cocartesian there is a unique morphism

$$V : W_n(B) \otimes W_n\Omega^i_A \to W_{n+1}(B) \otimes W_{n+1}\Omega^i_A$$

such that $V(FRx \otimes y) = x \otimes Vy$.

This defines a de Rham-V-procomplex and we use universality to get a mao inverse to the original one. $\square$

Definition 4.10. Let X be a ringed topos of $\mathbb{F}_p$-algebras. The complex

$$W\Omega_X := \varprojlim W_n\Omega_X$$

is called the de Rham–Witt complex of X. It is a differential graded algebra, with zero component $W(\mathscr{O}_X)$.

The maps V deine by passing to the limit an additive map V on $W\Omega_X$, which satisfies

$$\begin{aligned} xVy &= V(\mathrm{F}\, x.y) \quad \text{for } x \in W(\mathscr{O}_X), y \in W\Omega^i_X \\ (d[x])Vy &= V(([x]^{p-1}d[x])y) \quad \text{for } x \in \mathscr{O}_X, y \in W\Omega^i_X \\ V(xdy) &= Vx.dVy \quad \text{for } x \in W\Omega^i_X, y \in W\Omega^j_X \end{aligned}$$

4.3 An important example

In order to compare the hyper cohomology of the de Rham–Witt complex with crystalline cohomology, we look first at a basic example. We want to compute the de Rham–Witt complex of $X = (\mathbb{G}^r_a \times \mathbb{G}^s_m)_{\mathbb{F}_p}$. Thus let $A = \mathbb{F}_p[(T_i)_{1\leqslant i\leqslant n}, (T_i^{-1})_{i\in P}]$ where, $n = s + r$ and $P \subset \{1, \dots n\}$, $\#P = s$. (We will in particular need the cases when $s = 0$, i.e. $\mathbb{G}^n_a$, and $s = n$, i.e. $\mathbb{G}^n_m$).

We introduce now the rings

$$\begin{aligned} B &= \mathbb{Z}_p[(T_i)_{1\leqslant i\leqslant n}, (T_i^{-1})_{i\in P}] \\ C &= \bigcup_{r\geqslant 0} \mathbb{Q}_p[(T_i^{p^{-r}})_{1\leqslant i\leqslant n}, (T_i^{-p^{-r}})_{i\in P}] \end{aligned}$$

We have

$$d(T_i^{p^{-r}}) = p^{-r} T_i^{p^{-r}} \frac{dT_i}{T_i}$$

which shows that every form $\omega \in \Omega^m_{C/\mathbb{Q}_p}$ can be written uniquely as

$$\omega = \sum_{i_1<\dots<i_m} a_{i_1\dots i_m}(T) d\log T_{i_1} \dots d\log T_{i_m}$$

with $a_{i_1\ldots i_m}(T) \in C$ polynomials over $\mathbb{Q}_p$ in $T_i^{p^{-r}}$ and $T_i^{-p^{-r}}$ for $r \geqslant 0$, divisible by $\prod_{i_j \notin P} T_{i_j}^{p^{-s}}$ for some $s \in \mathbb{N}_0$.

Definition 4.11. We say ω is integral if its coefficients are polynomials over $\mathbb{Z}_p$.

Now we set

$$E_A^m = \left\{ \omega \in \Omega^m_{C/\,\mathbb{Q}_p} \,\middle|\, \omega \text{ and } d\omega \text{ are integral} \right\}$$

which gives a subcomplex $E_A^\bullet \subset \Omega_{C/\,\mathbb{Q}_p}$ (the biggest subcomplex consisting of integral forms). In particular, it is a sub-dga containing $\Omega_{B/\,\mathbb{Z}_p}$.

Example 4.12. $T_1^{\frac{1}{p}}$ does not belong to E^0 but $pT_1^{\frac{1}{p}}$ does.

We define two operators F and V on C: an automorphism

$$F(T_i^{p-r}) = T^{p^{-r+1}}$$

and an endomorphism

$$V = pF^{-1}$$

They extend to $\Omega_{C/\,\mathbb{Q}_p}$ (by acting on the coordinates: $F \sum a_{i_1\ldots i_m}(T) d\log T_{i_1} \ldots d\log T_{i_m} = \sum F a_{i_1\ldots i_m}(T) d\log T_{i_1} \ldots d\log$ and $V \sum a_{i_1\ldots i_m}(T) d\log T_{i_1} \ldots d\log T_{i_m} = \sum V a_{i_1\ldots i_m}(T) d\log T_{i_1} \ldots d\log T_{i_m}$), and one verifies

$$dF = pFd\,,\; Vd = pdV$$

so that in particular, $E^\bullet$ is stable by F and V. Furthermore, one has for $x, y \in \Omega_{C/\,\mathbb{Q}_p}$

$$\begin{aligned} xVy &= V(Fx.y) \\ V(xdy) &= (Vx)(dVy) \end{aligned}$$

The idea now is to set $E_n^m = E^m/(V^nE^m + dV^nE^{m-1})$ and to get a complex

$$\cdot \to E^\bullet_{n+1} \to E^\bullet_n \to E^\bullet_{n-1} \to \cdots$$

The identification $E^0/V^nE^0 \cong W_n(A)$ then induces a structure of V-procomplex $E^\bullet_\bullet$, and we will see that the induced morphism

$$W_\bullet\Omega_A \to E^\bullet_\bullet$$

is in fact an isomorphism.

We will start with the following proposition.

Proposition 4.13. *Keep all the notation from before.*

1. *E^0 is the set of elements $x = \sum a_k T^k \in C$ (using multi indices) such that $a_k \in \mathbb{Z}_p$ and the denominators of all k_i divide a_k.*
2. *We have the identities*

$$\begin{aligned} E^0 &= \sum_{n\in\mathbb{N}_0} V^n B \\ \bigcap_{n\in\mathbb{N}_0} V^n E^0 &= 0 \\ B \cap V^n E^0 &= p^n B \end{aligned}$$

3. *The homomorphism of $\mathbb{Z}_p$-algebras $B \to W(A)$ sending $T_i \mapsto [T_i]$ to its Teichmüller representative, extends in a unique way to a morphism of $\mathbb{Z}_p$-algebras*

$$\tau : E^0 \to W(A)$$

such that $\tau V = V\tau$, It is injective and induces for each $r \in \mathbb{N}$ an isomorphism

$$E^0/V^rE^0 \xrightarrow{\sim} W(A)/V^rW(A).$$

Proof. The first claim follows by definition: x has to be integral, so $a_k \in \mathbb{Z}_p$. For $dx = \sum k a_k T^k d\log T$ to be integral, the $ka_k \in \mathbb{Z}_p$. Note that k_i is of the form $\frac{k_i'}{p^{r_i}}$ with $k_i \in \mathbb{Z}$ and $r_i \in \mathbb{N}_0$, and $(k_i', p^{r_i}) = 1$. Thus the denominator has to divide a_k.

For the second claim, first identity: it is clear that $\sum V^n B \subset E^0$. On the other hand, let $x = aT^k \in E^0$, and p^s the biggest denominator of the k_i. Then we have just seen, that $p^s|a$ and thus we can write $aT^k = V^s p^{-s} a T^{p^s k}$ with $p^{-s} a T^{p^s k} \in B$.

Second and third identity : $x = \sum a_k T^k \in V^n E^0$ means $p^n | a_k$ for all k. Taking the limit over n induces $x = 0$. Also, then $B \cap V^n E^0 = p^n B$ is clear.

For the third claim: Existence of the morphism τ. Set

$$\begin{aligned} \overline{A} &= \bigcup_{r\geqslant 0} \mathbb{F}_p[(T_i^{p^{-r}})_{1\leqslant i\leqslant n}, (T_i^{-p^{-r}})_{i\in P}] \\ \overline{B} &= \bigcup_{r\geqslant 0} \mathbb{Z}_p[(T_i^{p^{-r}})_{1\leqslant i\leqslant n}, (T_i^{-p^{-r}})_{i\in P}] \end{aligned}$$

We have $E^0 \subset \overline{B}$ and F on $\overline{B}$ given by $T_i^{p^{-r}} \mapsto T_i^{p^{-r+1}}$ is an automorphism. Since $\overline{A}$ is perfect, The Witt vector Frobenius on $W(\overline{A})$ is also an automorphism. The morphism of $\mathbb{Z}_p$-algebras

$$\overline{B} \to W(\overline{A}) \, , \, T_i^{p^{-r}} \mapsto [T_i^{p^{-r}}]$$

is compatible with F and therefore with $V = pF^{-1}$. Thus the restriction to $E_0 = \sum_{n\in\mathbb{N}_0} V^n B$ induces the desired morphism τ (as it has image in $W(A)$). It is unique because of the identity $E^0 = \sum_{n\in\mathbb{N}_0} V^n B$.

Now to prove the isomorphism of the quotients mod V^r, note that V^r induces an A-linear homomorphism $F^r_* A \to V^r E^0 / V^{r+1} E^0$ and an A-linear iso $F^r_* A \xrightarrow{\sim} V^r W(A)/V^{r+1} W(A)$ and we get a commutative diagram

$$\begin{tikzcd} & F^r_* A \arrow[dl] \arrow[dr] & \\ V^r E^0 / V^{r+1} E^0 \arrow[rr, "gr_V"] & & V^r W(A)/V^{r+1} W(A). \end{tikzcd}$$

To show that $E^0/V^r E^0 \to W(A)/V^r W(A)$ is an isomorphism, it is enough to show that the horizontal morphism in this diagram gr_V is an isomorphism, hence that $F^r_* A \to V^r E^0/V^{r+1}E^0$ is an isomorphism. Since V is injective on E^0, it is enough to consider $r = 0$, i.e. we have to see that the inclusion $B \subset E^0$ induces an isomorphism $A = B/pB \xrightarrow{\sim} E^0/VE^0$, which follows form the first and third equality of the second claim: $E^0 = \sum_{n\in\mathbb{N}_0} V^n B$ and $B \cap V^n E^0 = p^n B$. Passing to the limit, we obtain an isomorphism

$$\varprojlim E^0/V^r E^0 \xrightarrow{\sim} W(A)$$

and composing with the canonical application $E^0 \to \varprojlim E^0/V^r E^0$ gives exactly τ. And because of the second equality from above, $\bigcap_{n\in\mathbb{N}_0} V^n E^0 = 0$, $E^0 \to \varprojlim E^0/V^r E^0$ is injective, and therefore τ is injective. □

Now we consider the filtration

$$\operatorname{Fil}^r E^i = V^r E^i + dV^r E^{i-1}$$

For each r, the $\operatorname{Fil}^r E^i$, $i \geqslant 1$ form a dgi of $\operatorname{Fil}^r E$ and we have

$$\operatorname{Fil}^0 E = E \supset \operatorname{Fil}^1 E \supset \cdots \supset \operatorname{Fil}^r E \supset \cdots$$

which gives a projective system of dga's

$$E_r = E/\operatorname{Fil}^r E$$

By definition we have $V(\operatorname{Fil}^r E) \subset \operatorname{Fil}^{r+1} E$ and $F \operatorname{Fil}^{r+1} E \subset \operatorname{Fil}^r E$, so that V induces an additive morphism, ad F a morphism of dga's

$$V : E_r \to E_{r+1} \text{ and } F : E_{r+1} \to E_r$$

satisfying the "usual" formulae

$$\begin{cases} dF = pFd, & Vd = pdV \\ xVy = V(Fx.y) & \text{for } x \in E_{r+1}, y \in E_r \\ V(xdy) = Vx.dVy & \text{for } x, y \in E_r \end{cases} \tag{4.2}$$

Theorem 4.14. *The projective system $E_\bullet$ with the operator V and the identification $E_r^0 \cong W_r(A)$ for $r \geqslant 1$ is a de Rham-V-procomplex. Moreover, the map*

$$W_\bullet \Omega_A \to E_\bullet$$

extending the identity of A is an isomorphism

In order to prove this, we have to study the structure of E. We will use the notion of basic Witt differentials, which was picked up by Langer and Zink later in their relative construction.

The ring C introduced above has a natural grading, of type

$$G = \left\{ k \in \mathbb{Z}[\frac{1}{p}]^n \mid k_i \geqslant 0 \text{ for } i \notin P \right\}$$

meaning, that the degree of an element is given by the multi-exponents of the variables, which are integers possibly divided by p, negative for $i \in P$, and positive for $i \notin P$ We can extend this grading to $\Omega_{C/\mathbb{Q}_p}$ by saying that a form has degree $k \in G$ if its coordinates are of this degree. Then $E \subset \Omega_{C/\mathbb{Q}_p}$ is a graded sub-complex. Denote the homogeneous component of degree k by ${}_k\Omega_{C/\mathbb{Q}_p}$ and similar or E.

We will use this to find a basis for E. Let $k \in G$ such that $\nu_p(k_1) \leqslant \cdots \leqslant \nu_p(k_n)$. Note that here if k_1 is an integer, so are all k_i, and if $k_r = 0$, then $k_{i \geqslant r} = 0$. Let I_m be the set of integer tuples ($\underline{i} = (i_1, \dots, i_m)$) such that $i_1 \leqslant \cdots \leqslant i_m$ and $k_{i_j} > 0$ for j such that $i_j \notin P$. Then we set

$$t_0 = \begin{cases} 1 & \text{if } i_i = 1 \\ p^{-\nu_p(k_1)} T^k_{[1,i_1[} & \text{if } i_i > 1 \text{ and } k_1 \notin \mathbb{Z}, \\ T^k_{[1,i_1[} & \text{if } i_1 > 1 \text{ and } k_1 \in \mathbb{Z} \end{cases}$$

and for $s \geqslant 1$

$$t_s = p^{-\nu_p(k_s)} T^k_{[i_s, i_{s+1}[}$$

Then we define

$$e_i(k) = t_0 \prod_{s \geqslant 1, k_{i_s} \neq 0} dt_s \prod_{s \geqslant 1, k_{i_s} = 0} d \log T_{i_s} \in_k \Omega^m_{C/\mathbb{Q}_p}$$

and

$$e_0(k) = \begin{cases} p^{-\nu_p(k_1)} T^k & \text{if } k_1 \notin \mathbb{Z}, \\ T^k & \text{otherwise} \end{cases}$$

Proposition 4.15. *Let $k \in G$ such that $\nu_p(k_1) \leqslant \cdots \leqslant \nu_p(k_n)$. For $m \in \mathbb{N}$, the $\mathbb{Z}_p$-module ${}_kE^m$ is free of finite type. The element $e_0(k)$ is a basis for ${}_kE^0$, and for $m \geqslant 1$, the elements $e_{\underline{i}}(k)$ for $\underline{i} \in I_m$ form a basis of ${}_kE^m$.*

Proof. This is a relatively technical proof, that involves juggling around with differentials. It is done by induction. For now I want to omit it. □

The general case, where k does not satisfy $\nu_p(k_1) \leqslant \cdots \leqslant \nu_p(k_n)$, can be deduced from this by applying permutations, as can be imagined easily. More precisely, for each k, we may choose a permutation σ_k, that reorders k, only if the above hypothseis is not satisfied. We denote with a prime the new objects.

Proposition 4.16. *E is generated by E^0 as $\mathbb{Z}_p dga$ (i.e. the $\mathbb{Z}_p$-dga morphism $\Omega_{E^0/\mathbb{Z}_p} \to E$ is surjective), and for each $r \geqslant 1$, Fil^r is a dgi of E generated by $V^r E^0$.*

Proof. The first claim follows directly after identifying a basis of the homogenous components in the previous proposition: we look at the homogenous components. For the integral components ($k_1 \in \mathbb{Z}$ and therefore all other $k_i \in \mathbb{Z}$) this is just a classical statement. For the case $k - 1 \notin \mathbb{Z}$, note that $de_{\underline{i}}(k) = e_{(1,\underline{i})}(k)$ and these elements generate ${}_kE^{m+1}$ as a $\mathbb{Z}_p$-module.

For the second claim, let I^r_E (or$I^r_{E^0}$) be the dgi generated by $V^r E^0$ in E (in E^0). Since $\mathrm{Fil}^r E^0 = I^r_{E^0} = V^r E^0$, th inclusion $\mathrm{Fil}^r E \supset I^r_E$ is clear. The other inlcusion follows from the fact, that E^0 generates E as $\mathbb{Z}_p$-algebra. □

We also need to know, what happens to the basic differentials, if we apply the operators V and F as well as the derivative d to them.

Proposition 4.17. *Let $k \in G$ and $k' = (k_{\sigma_k(i)})$ as described previously. For $m \in \mathbb{N}$ and $\underline{i} \in I_m$*

1. *If $1 < i_1$ or $m = 0$*
$$de_{\underline{i}}(k) = \begin{cases} p^{\nu_p(k'_1)} e_{(1,\underline{i})}(k) & \text{if } k'_1 \in \mathbb{Z} \\ e_{(1,\underline{i})}(k) & \text{if } k'_1 \notin \mathbb{Z} \end{cases}$$
If $i_1 = 1$,
$$de_{\underline{i}}(k) = 0$$
2. *If $1 < i_1$ or $m = 0$*
$$Ve_{\underline{i}}(k) = \begin{cases} pe_{\underline{i}}(\frac{k}{p}) & \text{if } \frac{k'_1}{p} \in \mathbb{Z} \\ e_{\underline{i}}(\frac{k}{p}) & \text{if } \frac{k'_1}{p} \notin \mathbb{Z} \end{cases}$$
If $i_1 = 1$,
$$Ve_{\underline{i}}(k) = pe_{\underline{i}}(\frac{k}{p})$$
3. *If $1 < i_1$ or $m = 0$*
$$Fe_{\underline{i}}(k) = \begin{cases} e_{\underline{i}}(pk) & \text{if } k'_1 \in \mathbb{Z} \\ pe_{\underline{i}}(pk) & \text{if } k'_1 \notin \mathbb{Z} \end{cases}$$
If $i_1 = 1$,
$$Fe_{\underline{i}}(k) = e_{\underline{i}}(pk$$

Proof. It is enough to show this for the reordered k. In this case, it just follows from the definition. □

Proposition 4.18. *Let $r \in \mathbb{N}$, $k \in G$. Set $s = s(k) = -\inf_{1 \leqslant i \leqslant n} \nu_p(k_i)$, and*
$$\nu(r,k) = \begin{cases} r - s & \text{if } s > 0, r \geqslant s \\ 0 & \text{if } s > 0, r < s \\ r & \text{if } s \leqslant 0 \end{cases}$$
Then
$${}_k\operatorname{Fil}^r E = p^{\nu(r,k)}({}_kE).$$

Proof. This is a bit tedious, but not hard. □

Corollary 4.19. *Multiplication by p induces a monomorphism $p : E_r \to E_{r+1}$. The components of*
$$\widehat{E} := \varprojlim E_r$$
are p-torsion free and the canonical map $E \to \widehat{E}$ is injective.

Proof. Since the ideal $\operatorname{Fil}^r E$ has a grading with respect to G, we have
$$E_r = \oplus_{k \in G}\, {}_kE_r.$$
For a chosen homogeneous component one verifies easily, that multiplication by p induces a monomorphism ${}_kE_r \to_k E_{r+1}$. The first claim follows. Hence, it is also true that $\widehat{E}$ is p-torsion free. Moreover, for each $k \in G$, $\bigcap_{r \in \mathbb{N}_0} {}_k\operatorname{Fil}^r E = 0$, so that the canonical map $E \to \widehat{E}$ is injective. □

We are now in a good position to proof the main theorem of this section. For the first part, we have to see, that the system $E_\bullet$ with V and $E^0_r = W_r(A)$ is a de Rham-V-procomplex. Since we have verified the formulae (4.2), the only point to verify form the definition of de Rham-V-procomplex is (V3) $(Vy)d[x] = V([x]^{p-1}y)d[x]$ for $x \in A$ and $y \in E^0_m$. It is sufficient to prove $Fd[x] = [x]^{p-1}d[x]$ because then
$$V([x]^{p-1}y)dV[x] = V([x]^{p-1}ydx) = V(yFd[x]) = d[x].Vy$$
First note, that by passing to the limit $F : E_r \to E_{r-1}$ defines an endomorphism of graded algebras on $\widehat{E}$ such that $dF = pFd$. With $F[x] = [x]^p$ we have $pFd[x] = dF[x] = p[x]^{p-1}d[x]$. As E^1 is p-torsion free, we can divide by p, and get the desired equality.

By the universal property of $W_\bullet\Omega_A$, this means that the identity on A now extends to a morphism of de Rham-V-pro complexes

$$\phi_\bullet : W_\bullet\Omega_A \to E_\bullet$$

and we have to show, that it is in fact an isomorphism. We will construct an inverse to this, by sending the base elements $e_i(k)$ of $E_\bullet$ to certain elements of $W_\bullet\Omega_A$.

We consider again the case $k \in G$ with $\nu_p(k_1) \leqslant v_2 \leqslant \cdots \leqslant \nu_p(k_n)$ — more general cases follow again with permutations. Let $f_0(k) \in W(A)$ be

$$f_0(k) = \begin{cases} p^{-\nu_p(k_1)}[T]^k & \text{if } k_1 \notin \mathbb{Z} \\ [T]^k & \text{if } k_1 \in \mathbb{Z} \end{cases}$$

For $m \geqslant 1$ and $\underline{i} \in I_m$

$$y_0 = \begin{cases} 1 & \text{if } i_1 = 1 \\ p^{-\nu_p(k_1)}[T]^k_{[1,i_1[} & \text{if } i_1 > 1 \text{ and } k_1 \notin \mathbb{Z} \\ [T]^k_{[1,i_1[} & \text{if } i_1 > 1 \text{ and } k_1 \in \mathbb{Z} \end{cases}$$

For $s \geqslant 1$ such that $v_p(i_s) < 0$

$$y_s = p^{-\nu_p(k_{i_s})}[T]^k_{[i_s,i_{s+1}[}$$

and for $s \geqslant 1$ such that $0 \leqslant \nu_p(k_{i_s}) < \infty$

$$z_s = [T]^{p^{-\nu_p(k_{i_s})}k}_{[i_s,i_{s+1}[}$$

Now set $f_{\underline{i}}(k) \in W\Omega^m_A$ to be

$$f_{\underline{i}}(k) = y_0 \prod_{s\geqslant 1,\nu_p(k_{i_s})<0} dy_s \prod_{s\geqslant 1, 0\leqslant\nu_p(k_{i_s})<\infty} z_s^{p^{\nu_p(k_{i_s})}-1}dz_s \prod_{s\geqslant 1,\nu_p(k_{i_s})=\infty} d\log[T_{i_s}].$$

Now we define a map $E_\bullet \to W_\bullet\Omega_A$ by sending

$$e_i(k) \mapsto f_i(k)$$

One verifies without difficulty that this commutes with d and V. It is compatible with the filtration on both sides if we define a filtration

$$\mathrm{Fil}'^r\, W\Omega_A = V^r W\Omega_A + dV^r W\Omega_A^{\bullet-1}$$

which is contained in $\ker(W\Omega_A \to W_r\Omega_A$. Thus, we defined a projective system of morphism of complexes

$$\psi_\bullet E_\bullet \to W_\bullet\Omega_A$$

By definition, $\phi_\bullet\psi_\bullet = \mathrm{id}$, hence it is sufficient, to show that $\psi_\bullet$ is surjective.

Consider the injection $B \subset E^0 \subset W(A)$, which extends to a morphism of $\mathbb{Z}_p$-dga's $\Omega_B \to \Omega_{W(A)}$ which together with the canonical projection gives

$$\Omega_B \to W\Omega_A$$

and this in turn is just the restriction of ψ as they coincide on the base elements $e_i(k)$ for $k \in G \cap \mathbb{Z}^n$.

Let $M \subset W\Omega_A$ be the sub-$\mathbb{Z}_p$-dga generated by $[T]^k$ for $k \in G \cap \mathbb{Z}^n$, $M_\bullet$ its image in $W_\bullet\Omega_A$. Then

$$\psi_\bullet(E_\bullet) \supset M_\bullet$$

Since $\psi_\bullet$ is compatible with V, the subjectivity results form the following identity

$$W_j\Omega^i_A = \sum_{0\leqslant r<j} V^r M^i_{j-r} + \sum_{0\leqslant r<j} dV^r M^{i-1}_{j-r}$$

This need some computation to verify, the interested reader should do it as an exercise.

This finishes the proof of the main theorem.

4.4 The endomorphism F on $W\Omega$

The Frobenius on $E_\bullet$ induces a Frobenius morphism on $W_\bullet\Omega_A$-

Theorem 4.20. *Let X be a ringed topos of $\mathbb{F}_p$-algebras. The homomorphism of projective systems $RF = FR : W_\bullet \mathscr{O}_X \to W_{\bullet-1} \mathscr{O}_X$ extends uniquely to a morphism of projective systems of graded algebras*

$$F : W_\bullet\Omega_X \to W_{\bullet-1}\Omega_X$$

such that for $x \in \mathscr{O}_X$

$$Fd[x] = [x]^{p-1}d[x]$$

and

$$FdV = d : W_n \mathscr{O}_X \to W_n\Omega^1_X$$

In particular, $Fd : W_n \mathscr{O}_\to W_{n-1}\Omega^1_X$ is given by the formula

$$Fdx = [x_0]^{p-1}d[x_0] + d[x_1] + \ldots + dV^{n-2}[x_{n-1}]$$

Uniqueness follows from the fact, that an element $x \in W_n \mathscr{O}_X$ can be written as

$$x = [x_0] + V[x_1] + \ldots + V^{n-1}[x_{n-1}]$$

(and from subjectivity of the projection $\Omega_{W_n \mathscr{O}_X} \to W_n\Omega_X$). The uniqueness also implies, that for a morphism of topoi $f : X \to Y$, the induced morphism

$$W_\bullet\Omega_Y \to f_*W_\bullet\Omega_X$$

is compatible with F. We can pass to limits to get a homomorphism of graded algebras

$$F : W\Omega_X \to W\Omega_X$$

satisfying the usual equalities. Note however, that this endomorphism, since it is an endomorphism of complexes, coincides with p^iF in degree i. It would be a useful exercise to show this explicitly.

4.5 Comparison with crystalline cohomology

During this section, let S be a perfect scheme of characteristic $p > 0$ - e.g. $S = \operatorname{Spec} k$ as before. Let $f : X \to S$ be a an S-scheme of finite type. Let $u_n : (X/W_n(S))_{\text{cris}} \to X_{\text{zar}}$ be the canonical projection of topoi. We will define a morphism

$$Ru_n(\mathscr{O}_{X/W_n}) \to W_n\Omega_X \tag{4.3}$$

and show that it is a quasi-isomorphism in case f is smooth. By applying Rf_* and $R\Gamma(X, -)$ to this morphism, one obtains morphisms

$$Rf_{X/W_n}(\mathscr{O}_{X/W_n}) \to Rf_*(W_n\Omega_X)$$

with $f_{X/W_n} = f \circ u_{X/W_n} : (X/W_n)_{\text{cris}} \to (W_n)_{\text{zar}}$, as well as

$$\begin{array}{rcl} R\Gamma_{\text{cris}}(X/W_n) & \to & R\Gamma(X, W_n\Omega) \\ H^\bullet_{\text{cris}}(X/W_n) & \to & H^\bullet(X, W_n\Omega) \end{array}$$

which are also isomorphisms in case X/S is smooth.

Let us start by constructing the morphism (4.3). Assume first, that there is a closed immersion $X \hookrightarrow Y$ in a formal smooth schemes over W endowed with a Frobenius lift $F : Y \to Y^\sigma = Y \times_\sigma W$. For $Y_n = Y \times W_n$ let $\overline{Y}_n$ be the PD-envelope of X in Y_n. In this setup, recall Berthelot's comparison theorem

Theorem 4.21. *There is a canonical quasi-isomorphism*

$$Ru_n(\mathscr{O}_{X/W_n}) \xrightarrow{\sim} \mathscr{O}_{\overline{Y}_n} \otimes \Omega_{Y_n/W_n} = \Omega_{\overline{Y}_n/W_n,[-]}$$

where on the right hand side, we find the PD-de Rham complex.

This sets us up to construct a morphism from the PD-de Rham complex on the right hand side to the de Rham-Witt complex.

From the existence of a Frobenius lift, it follows, that the closed immersion $X \hookrightarrow Y$ extends to an immersion $W_n(X) \hookrightarrow Y$. Namely, let

$$\mathscr{O}_Y \xrightarrow{t_F} W_(\mathscr{O}_{Y_1}) \to i_{1*}W_n(\mathscr{O}_X)$$

where the second arrow is by functoriality given by $i_1 : X \hookrightarrow Y_1$. It sends the ideal $p^n\mathscr{O}_Y$ into $i_{1*}V^nW(\mathscr{O}_X)$ and induces a morphism

$$\mathscr{O}_{Y_n} \to i_{1*}W_n(\mathscr{O}_X). \tag{4.4}$$

Thus, we want to factor $X \to \overline{Y}_n$ through $W_n(X)$. The morphism (4.4) sends the ideal of $X \hookrightarrow Y_n$ to $i_{1*}VW_{n-1}(\mathscr{O}_X)$, which has a natural PD-structure given by

$$\gamma_n(Vx) = \frac{p^{n-1}}{n!}V(x^n)$$

Hence, we can consider the induced PD-morphism

$$\mathscr{O}_{\overline{Y}_n} \to W_n(\mathscr{O}_X).$$

This induces a morphism of de Rham complexes

$$\Omega_{\overline{Y}_n} \to \Omega_{W_n\mathscr{O}_X} \xrightarrow{\pi_n} W_n\Omega_X$$

factoring through the PD-de Rham complex $\Omega_{\overline{Y}_n,[-]} = \Omega_{\overline{Y}_n}/(d\gamma_k(x) = \gamma_{k-1}(x)dx)$.

$$\begin{tikzcd}
\mathscr{O}_{\overline{Y}_n} \otimes \Omega_{Y_n/W_n} \arrow[r] \arrow[drrr] & \Omega_{\overline{Y}_n/W_n,[-]} \arrow[r] \arrow[drr] & \Omega_{W_n(\mathscr{O}_X)}/(d\gamma_k(Vx) - \gamma_{k-1}(Vx)dx) \arrow[dr] & \\
Ru_n\mathscr{O}_{X/W_n} \arrow[u, "\sim"] \arrow[rrr] & & & W_n\Omega_X
\end{tikzcd}$$

One shows that this construction is independent of choices (of Y and F), by considering for two different Y, Y' with Frobenius lifts F, F' the product $(i, i')X \hookrightarrow Z = Y \times_W Y'$ and $G = F \times_W F'$ to get diagrams

$$\begin{tikzcd}
Ru_n\mathscr{O}_{X/W_n} \arrow[r, "\sim"] \arrow[d, equal] & \Omega_{\overline{Y}_n/W_n,[-]} \arrow[r] \arrow[d] & W_n\Omega_X \arrow[d] \\
Ru_n\mathscr{O}_{X/W_n} \arrow[r, "\sim"] & \Omega_{\overline{Z}_n/W_n,[-]} \arrow[r] & W_n\Omega_X
\end{tikzcd}$$

In general, we can't assume the existence of a closed immersion $X \hookrightarrow Y$ factoring through $W_r(X)$ globally, but only locally. Then one uses a descent argument with respect to an appropriate covering. This will be an exercise.

We come to the main result of this section.

Theorem 4.22. *The morphism (4.3) is a quasi-isomorphism.*

Proof. Because this is a local question, we may assume that X and S are affine – $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} k$ – and choose a flat p-adically complete lift B of A over $W(k)$, together with a Frobenius lift F compatible with σ.

To define the comparison morphism as above, use the immersion of X in the formal scheme $Y = \operatorname{Spf}(B)$ together with F. The ideal of $B_r \to A$ is pB_r, which has a natural PD-structure extending the canonical one. Thus we don't have to modify it to obtain the PD-envelope: $\overline{B}_n = B_n$ and

$$Ru_r\mathscr{O}_{X/W_n} \xrightarrow{\sim} \Omega_{B_r}.$$

Using t_F as above, we obtain a morphism $B_n \to W_r(A)$ so

$$\Omega_{B_r} \to W_r\Omega_A,$$

which we have to show is a quasi-isomorphism. It is the same to take the limit on both sides

$$\Omega_B \to W\Omega_A,$$

and show that it induces a quasi-isomorphism on graded pieces for the $p$adic filtration on Ω_B and the canonical filtration on $W\Omega_A$

$$\operatorname{Fil}^r W\Omega_X = \begin{cases} W\Omega_X & \text{if } r \leqslant 0 \\ \ker(W\Omega_X \to W_r\Omega_X) & \text{if } r \geqslant 1 \end{cases}$$

The question is local, so by étale localisation we may reduce to the case, when $A = \mathbb{F}_p[\underline{T}]$, $B = \mathbb{Z}_p[\underline{T}]$ and $C = \mathbb{Q}_p[\underline{T}]$ (to see this, let A be étale over $\mathbb{F}_p[\underline{T}]$, then by functoriality there is an isomorphism $W_rA \otimes \operatorname{Fil}^n W_r\Omega_{\mathbb{F}_p[\underline{T}]} \xrightarrow{\sim} \operatorname{Fil}^n W_r\Omega_A$, so it is enough to consider $A = \mathbb{F}_p[\underline{T}]$).

So we can consider the complex $E^\bullet_\cdot$ defined earlier: we have to show that $\Omega_B/p^n \to E^\bullet_n$ is a quasi-isomorphism. We know that there is an injection

$$\Omega_B \hookrightarrow E^\bullet \hookrightarrow \Omega_{C/\mathbb{Q}_p}$$

Recalling the grading G introduced earlier, we note, that Ω_B consists exactly of thus forms in $E^\bullet$ that have integral weight. Thus we have for each r

$$E^\bullet_r \cong \Omega_{B_r} \oplus \bigoplus_{g\in G, g\notin\mathbb{Z}^n} {}_gE^\bullet_r$$

Delgine showd that for $g \notin \mathbb{Z}^n$ the complex ${}_gE_r$ is homotopically trivial. It follows that the inclusion $\Omega_B \hookrightarrow E$ is a homotopy equivalence, and for each r the inclusion $p^r\Omega_B \hookrightarrow \operatorname{Fil}^r E$ is a homotopy equivalence, such that

$$\Omega_{B_r} = \Omega_B/p^r\Omega_B \hookrightarrow E_r$$

is a quasi-isomorphism. □

It remains to show Deligne's result.

Proposition 4.23. *For $g \notin \mathbb{Z}^n$, the complex ${}_gE$ is homotopically trivial.*

Proof. Wlog we may assume that $g_1 \notin \mathbb{Z}$ (thus $g_1^{-1} \in \mathbb{Z}$). We have to find a homotopy. For this, let h be the operator on $\Omega_{C/\mathbb{Q}_p}$ given by the inner product with $g_1^{-1}T_1\frac{d}{dT_1}$: for $x = \sum_{i_1<\ldots<i_m} a_{i_1,\ldots,i_m}(T) d\log T_{i_1}\cdots d\log T_{i_m} \in \Omega_C^m$

$$hx = g_1^{-1}\sum_{i_1<\ldots<i_m} a_{i_1,\ldots,i_m}(T) d\log T_{i_2}\cdots d\log T_{i_m}.$$

In particular, if x is an integral (i.e. has integral coefficients) form, hx is also integral, and h preserves the weight (homogenous degree) g, which is measured solely on the coefficients. With this definition, the commutator

$$\theta_{g_1^{-1}T_1\frac{d}{dT_1}}S = dh + hd$$

can be seen as the Lie derivative (using the notation of Cartan, nowadays often denoted by $\mathscr{L}_{g_1^{-1}T_1\frac{d}{dT_1}}$, "Cartan's magic formula"). Hence, if x is of weight g

$$(dh + hd)(x) = x$$

This is obviously true for function $a(T)$, and because of $d\theta_X\omega = \theta_X d\omega$ with a form ω and a vector field X, this is true in general. Moreover, since by hypothesis dx is integral, hdx is by the above reasoning also integral and so is $dhx = x - hdx$. Thus indeed $hx \in {}_gE$ and h gives a homotopy on ${}_gE$ between the identity and the zero map. □

5 The big de Rham–Witt complex

In this section we will introduce the big de Rham–Witt complex following Lars Hesselholt's paper [7] in Section 4. The original definition is due to Hesselholt and Madsen in [8] which relies on the adjoint functor theorem. However, there was an issue with 2-torsion. This was solved by Lars Hesselholt using λ-ring theory.

We will see how this construction generalises the p-typical de Rham–Witt complex from $\mathbb{F}_p$-algebras to $\mathbb{Z}_{(p)}$-algebras. At the end, we want to draw the relation to K-theory.

5.1 Big Witt complexes

Let S be a truncation set (recall that a truncation set is a subset $S \subset \mathbb{N}$ such that if $n \in S$ and $d|n$ then also $d \in S$). We will define the de Rham–Witt complex $\mathbb{W}\,\Omega_S$.

Let $\mathscr{J}$ be the set of truncation sets, partially ordered for inclusion. We consider it as a category with a morphism from T to S if $T \subset S$.It is clear that the assignment

$$S \mapsto \frac{S}{n}$$

is an endofunctor of $\mathscr{J}$. And since $\frac{S}{n} \subset S$ there is a morphism from $\frac{S}{n}$ to S.

Recall that we defined a ring functor for each truncation set S

$$A \mapsto \mathbb{W}_S(A),$$

called the big Witt vectors. Now, instead of fixing S, we fix a ring A to get a contravariant functor

$$\begin{array}{ccc} \mathscr{J} & \to & \mathcal{A}nn \\ S & \mapsto & \mathbb{W}_S(A) \end{array}$$

from $\mathscr{J}$ to the category of rings, sending colimits to limits. Recall that we defined Frobenius and Verschiebung for any $n \in \mathbb{N}$

$$\begin{array}{rcl} F_n : \mathbb{W}_S(A) & \to & \mathbb{W}_{\frac{S}{n}}(A) \\ V_n : \mathbb{W}_{\frac{S}{n}}(A) & \to & \mathbb{W}_S(A) \end{array}$$

where the former is a ring homomorphism and the latter is additive (a morphism of abelian groups). These deine in fact natural transformations with respect to the "variable" S.

We will now consider the category of big Witt complexes. The de Rham-Witt complex for a truncation set S can then be defined as the initial object in this category.

Remark 5.1. This is reminiscent of the category of de Rham-V-procomplexes, whose initial object was the p-typical de Rham–Witt complex. One difference is, that here we need from the beginning a Frobenius, whereas in the p-typical case, the Frobenius came out of an explicit construction after having established the existence of an initial object. It should be remarked however, that in the case of the p-typical de Rham–Witt complex, one can also adopt a similar approach. In fact, there is a forgetful functor from the category of de Rham-V-procomplexes to the category of Witt complexes, simply forgetting the Frobenius. The de Rham–Witt complex can be defined as the initial object in either of them.

As mentioned above, the original definition of big Witt complexes due to Hesselholt and Madsen had an issue with 2-torsion. The first correct 2-typical definition for a Witt complex was given by Costeanu.

Definition 5.2. A (big) Witt complex over A is a contravariant functor

$$S \mapsto E_S^\bullet$$

assigning to every subtruncation set of U an anti-symmetric graded ring $E_S^\bullet$ that takes colimits to limits together with a natural ring homomorphism

$$\eta_S : \mathbb{W}_S(A) \to E_S^0$$

and natural maps of graded abelian groups

$$\begin{array}{rl} d: & E_S^r \to E_S^{r+1} \\ F_n: & E_S^r \to E_{\frac{S}{n}}^r \\ V_n: & E_{\frac{S}{n}}^r \to E_S^r \end{array}$$

such that

1. For $x \in E_S^r$, $y \in E_S^t$

$$\begin{array}{rcl} d(x \cdot y) & = & d(x) \cdot y + (-1)^r x \cdot d(y) \\ d(d(x)) & = & d\log \eta_S([-1]_S) \cdot d(x) \end{array}$$

2. For $m, n \in \mathbb{N}$

$$\begin{aligned}
F_1 &= V_1 = \mathrm{id} \\
F_m F_n &= F_{nm} \\
V_n V_m &= V_{mn} \\
F_n V_n &= n \cdot \mathrm{id} \\
F_m V_n &= V_n F_m \quad \text{if } (m,n) = 1 \\
F_n \eta_S &= \eta_{\frac{S}{n}} F_n \\
\eta_S V_n &= V_n \eta_{\frac{S}{n}}
\end{aligned}$$

3. For all $n \in \mathbb{N}$ the map F_n is a ring homomorphism and F_n and V_n satisfy the projection formula for $x \in E_S^r$ and $y \in E_{\frac{S}{n}}^t$
$$x \cdot V_n(y) = V_n(F_n(x)y).$$

4. For all $n \in \mathbb{N}$ and $y \in E_{\frac{S}{n}}^r$
$$F_n dV_n(y) = d(y) + (n-1) d \log \eta_{\frac{S}{n}}([-1]_{\frac{S}{n}}) \cdot y.$$

5. For all $n \in \mathbb{N}$ and $a \in A$
$$F_n d\eta_S([a]_S) = \eta_{S/n}([a]_{\frac{S}{n}}^{n-1}([a]_{\frac{S}{n}}).$$

A map of Witt complexes is a map of graded rings $f : E_S^\bullet \to \tilde{E}_S^\bullet$ such that

$$\begin{aligned}
f\eta_S &= \tilde{\eta} \\
fd &= \tilde{d}f \\
fF_n &= \tilde{F}_n f \\
fV_n &= \tilde{V}_n f.
\end{aligned}$$

Part of the structure of a Witt complex is a restriction map
$$R_T^S : E_S^\bullet \to E_T^\bullet$$
for $T \subset S$.

Lemma 5.3. *Every Witt complex is determined, up to canonical isomorphism , on finite truncation sets.*

Proof. For every truncation set S and $r \in \mathbb{N}$ the restriction maps define a bijection
$$E_S^r \to \varprojlim_{T \subset S, \text{ finite}} E_T^r$$
□

In particular, it follows from this that for $a \in \mathbb{W}(A)$ written as a convergent sum $a = \sum_{n \in S} V_n([a_n]_{\frac{S}{n}})$ the element $d\eta_S(a) \in E_S^1$ has a similar representation
$$d\eta_S(a) = \sum_{n \in S} dV_n([a_n]_{\frac{S}{n}}).$$

Remark 5.4. The issue with 2-torsion lies in the appearance of the element $d \log \eta_S([-1]_S)$. This element is annihilated by 2. Indeed, since d is a derivation

$$\begin{aligned}
2d \log \eta_S([-1]_S) &= \frac{d\eta_S([-1]_S)}{\eta_S[-1]_S} + \frac{d\eta_S([-1]_S)}{\eta_S[-1]_S} \\
&= \frac{\eta_S([-1]_S)}{\eta_S([1])} d\eta_S([-1]_S) + \frac{\eta_S([-1]_S)}{\eta_S([1])} d\eta_S([-1]_S) \\
&= \frac{d\eta_S([-1]_S[-1]_S)}{\eta_S([1]_S)} = d \log \eta_S([1]_S) = 0
\end{aligned}$$

It follows that $d\log\eta_S([-1]_S)$ is zero if 2 is invertible or i $2=0$ in A because then $[-1]_S=[1]_S$.

Moreover, since

$$[-1]_S=-[1]_S+V_2([1]_{\frac{S}{2}})$$

it follows that $d\log\eta_S([-1]_S)$ is also zero if S contains only odd integers.

We see therefore that in these cases, d is a differential and makes $E_S^\bullet$ into an anitsymmetric differential graded ring.

Lemma 5.5. *Let $m,n\in\mathbb{N}$, and $c=(m,n)$ the greatest common divisor, choose any pair $i,j\in\mathbb{Z}$ such that $mi+nj=c$. The following relations hold for every (big) Witt complex:*

$$\begin{aligned}
dF_n &= nF_nd\\
V_nd &= ndV_n\\
F_mdV_n &= idF_{\frac{m}{c}}V_{\frac{n}{c}}+jF_{\frac{m}{c}}V_{\frac{n}{c}}d+(c-1)d\log\eta_{\frac{S}{m}}([-1]_{\frac{S}{m}})\cdot F_{\frac{m}{c}}V\frac{n}{c}\\
d\log\eta_S([-1]_S) &= \sum_{r\in\mathbb{N}}2^{r-1}dV_{2^r}\eta_{\frac{S}{2^r}}([1]_{\frac{S}{2^r}})\\
d\log\eta_S([-1]_S)\cdot d\log\eta_S([-1]_S) &= 0\\
dd\log\eta_S([-1]_S) &= 0\\
F_n(d\log\eta_S([-1]_S)) &= d\log\eta_{\frac{S}{n}}([-1]_{\frac{S}{n}})
\end{aligned}$$

Proof. This follows mostly by explicit calculations. We will do some, and leave the rest as exercise. For the first equation:

$$\begin{aligned}
dF_n(x) &= F_ndV_nF_n(x)-(n-1)d\log\eta[-1]\cdot F_n(x) && \text{this follows from (4) of the definition}\\
&= F_nd(V_n\eta([1])\cdot x)-(n-1)d\log\eta([-1])\cdot F_n(x) && \text{from the projectin formula}\\
&= F_n(dV_n\eta([1])\cdot x+V_n\eta([1])\cdot dx)-(n-1)d\log\eta([-1])\cdot F_n(x) && \text{because } d \text{ is a derivation}\\
&= F_ndV_n\eta([1])\cdot F_n(x)+F_nV_n\eta([1])\cdot F_nd(x)-(n-1)d\log\eta([-1])\cdot F_n(x)\\
&= (n-1)d\log\eta([-1])\cdot F_n(x)+nF_nd(x)-(n-1)d\log\eta([-1])\cdot F_n(x) && \text{from (4) and (2) of the definition}\\
&= n_nd(x)
\end{aligned}$$

The calculation or the second equality is similar and left as an exercise.

Next we proof the last formula.

$$\begin{aligned}
F_n(d\log\eta_S([-1]_S)) &= F_n(\eta_S([-1]_S^{-1})d\eta_S([-1]_S)\\
&= F_n\eta_S([-1]_S^{-1})F_nd\eta_S([-1]_S)\\
&= \eta_{\frac{S}{n}}([-1]_{\frac{S}{n}}^{-n})\eta_{\frac{S}{n}}([-1]^{n-1})d\eta_{\frac{S}{n}}([-1]_{\frac{S}{n}} && \text{from (5) of the definition}\\
&= \eta_{\frac{S}{n}}([-1]_{\frac{S}{n}}^{-1})d\eta_{\frac{S}{n}}([-1]_{\frac{S}{n}})=d\log\eta_{\frac{S}{n}}([-1])
\end{aligned}$$

Using the three formulae already proved, we can compute the remaining equalities.

$$\begin{aligned}
F_mdV_n(x) &= F_{\frac{m}{c}}F_cdV_cV_{\frac{n}{c}}(x)\\
&= F_{\frac{m}{c}}dV_{\frac{n}{c}}(x)+(c-1)d\log\eta_{\frac{S}{c}}([-1]_{\frac{S}{c}})\cdot F_{\frac{m}{c}}V_{\frac{n}{c}}(x) && \text{with property (4) from the definition}\\
&= ((\frac{m}{c})i+(\frac{n}{c})j)F_{\frac{m}{c}}dV_{\frac{n}{c}}(x)+(c-1)d\log\eta_{\frac{S}{c}}([-1]_{\frac{S}{c}})\cdot F_{\frac{m}{c}}V_{\frac{n}{c}}(x)\\
&= idF_{\frac{m}{c}}V_{\frac{n}{c}}(x)+jF_{\frac{m}{c}}V_{\frac{n}{c}}(x)+(c-1)d\log\eta_{\frac{S}{c}}([-1]_{\frac{S}{c}})\cdot F_{\frac{m}{c}}V_{\frac{n}{c}}(x)
\end{aligned}$$

The sum formula for $d\log\eta_S([-1]_S)$ follows by induction: We know from an exercise that $[-1]_S=-[1]_s+V_2([1]_{\frac{S}{2}})$. Use this to show that

$$d\log\eta_S([-1]_S)=dV_2\eta_{\frac{S}{2}}([1]_{\frac{S}{2}})+V_2(d\log\eta_{\frac{S}{2}}([-1]_{\frac{S}{2}}))$$

then the induction argument is obvious.

Using this, we also find

$$\begin{aligned} dV_2(d\log\eta_{\frac{S}{2}}([-1]_{\frac{S}{2}}) &= \sum_{r\in\mathbb{N}} 2^r ddV_{2^{r+1}}\eta_{\frac{S}{2^{r+1}}}([1]_{\frac{S}{2^{r+1}}}) \\ &= \sum_{r\in\mathbb{N}} 2^r d\log\eta_S([-1]_S)\cdot dV_{2^{r+1}}\eta_{\frac{S}{2^{r+1}}}([1]_{\frac{S}{2^{r+1}}}) \quad \text{because of (1) of the definition} \\ &= 0 \quad \text{because } d\log\eta([-1]) \text{ is annihilated by } 2 \end{aligned}$$

With the equality $[-1]_S = -[1]_S + V_2([1]_{\frac{S}{2}})$ one can show (and the reader s encouraged to do this as an exercise)

$$(d\log\eta_S([-1]_S))^2 = dV_2(d\log\eta_{\frac{S}{2}}([-1]_{\frac{S}{2}}))\cdot\eta_S([1]_S - V_2([1]_{\frac{S}{2}})) = 0,$$

which is zero because the first factor is zero by what we just showed.

It follows from this that $(d\eta_S([-1]_S))^2 = 0$ if spell $d\log$ out. As an exercise, use this to show the last equality □

The next proposition wil play an important role in the λ-ring approach to the construction of the big de Rham–Witt complex.

Proposition 5.6. *For every Witt complex $E_S^\bullet$ over A and every $n\in\mathbb{N}$ the diagram*

$$\begin{tikzcd} \Omega^1_{\mathbb{W}_S(A)} \arrow[r, "\eta_S"] \arrow[d, "F_n"] & E^1_S \arrow[d, "F_n"] \\ \Omega^1_{\mathbb{W}_{\frac{S}{n}}(A)} \arrow[r, "\eta_{\frac{S}{n}}"] & E^1_{\frac{S}{n}} \end{tikzcd}$$

commutes

Proof. Wlog we can assume that $S = \mathbb{N}$, as the restriction map $R^{\mathbb{N}}_S$ commutes with Frobenius and the map η. Moreover, because a Witt complex is determined on finite truncation sets, and in particular we have a representation for $a\in\mathbb{W}(A)$

$$d\eta_S(a) = \sum_{n\in S} dV_n([a_n]_{\frac{S}{n}})$$

it is enough to show for every $n\in\mathbb{N}$, $p\in\mathbb{N}$ prime and $a\in A$

$$F_p dV_n\eta_{\mathbb{N}}([a]_{\mathbb{N}}) = \eta_{\mathbb{N}}F_p dV_n([a]_{\mathbb{N}})$$

in $E^1_{\mathbb{N}}$.

Case p does not devide n. Set $k = \frac{(1-n^{p-1})}{p}$ and $l = n^{p-2}$. Then $kp + ln = 1$, and $c = (p,n) = 1$ and F_p and V_n commute. Then by the previous lemma

$$\begin{aligned} F_p dV_n\eta([a]) &= k\cdot dV_nF_p\eta([a]) + l\cdot V_nF_pd\eta([a]) \\ &= k\cdot dV_n\eta([a]^p) + l\cdot V_n\eta([a]^{p-1}d[a]) \end{aligned}$$

Now we have to compute $\eta F_p dV_n([a])$. For this we need the equalities

$$F_p db = b^{p-1}db + d\left(\frac{F_p(b) - b^p}{p}\right)$$

and

$$V_m(a)^n = m^{n-1}V_m(a^n)$$

which are left to the reader as exercise.

$$\begin{aligned} \eta F_p dV_n([a]) &= \eta(V_n([a])^{p-1}\cdot dV_n([a]) + d\left(\frac{F_pV_n[a] - (V_n[a])^p}{p}\right)) \\ &= \eta(n^{p-2}\cdot V_n([a]^{p-1})\cdot dV_n([a]) + d\left(\frac{V_n([a]^p) - n^{p-1}V_n([a]^p)}{p}\right)) \\ &= \eta(l\cdot V_n([a]^{p-1})dV_n([a]) + kdV_n([a]^p)) \\ &= l\cdot V_n\eta([a]^{p-1})dV_n\eta([a]) + k\cdot dV_n\eta([a]^p) \\ &= l\cdot V_n\left(\eta([a]^{p-1})\cdot F_ndV_n\eta([a])\right) + k\cdot dV_n\eta([a]^p) \quad \text{because of the projection formula} \\ &= l\cdot V_n\eta([a]^{p-1}d[a]) + k\cdot dV_n\eta([a]^p) \quad \text{because of (4) if the definition and } n^{p-2}(n-1)d\log\eta([-1]) = 0 \end{aligned}$$

Case p divides n. In this case, one treats $p = 2$ and p odd separately. This will b done in the exercise session. ☐

In order to extend this diagram – and in particular the morphism η to complexes, we have to modify the usual complex Ω.

Remark 5.7. Note that the Frobenius $F_n : \Omega^1_{\mathbb{W}_S(A)} \to \Omega^1_{\mathbb{W}_{\frac{S}{n}}(A)}$ is not the one following from functoriality, but it is off by a constant factor. We will discuss the existence of such a Frobenius later on.

5.2 Two anticommutative graded algebras

The big de Rham–Witt complex is closely related to K-theory. In fact, it was introduces by Hesselholt and Madsen in order to give an algebraic description of the equivariant homotopy groups in low degrees of Bökstedt's topological Hochschild spectrum of a commutative ring. This functorial algebraic description is essential to understand algebraic K-theory by means of the cyclotomic trace map of Bökstedt–Hsiang–Madsen. Recall that for a field an easy description of Quillen K-theory up to degree 2 is given by Milnor K-theory. Therefore, we should not necesserily expect the big de Rham–Witt complex to be made up of alterating forms, but rather some sort of Steinberg relation should be saitsfied. This leads to the following definition.

Definition 5.8. Let A be a ring. The graded $\mathbb{W}(A)$-algebra

$$\widehat{\Omega}_{\mathbb{W}(A)} := T_{\mathbb{W}(A)}\Omega^1_{\mathbb{W}(A)}/J$$

is the quotient of the tensor algebra of the $\mathbb{W}(A)$-module $\Omega^1_{\mathbb{W}(A)}$ by the graded ideal generated by the elements of the form

$$da \otimes da - d\log[-1] \otimes F_2(da)$$

for $a \in \mathbb{W}(A)$.

The defining relation $da \cdot da = d\log[-1] \cdot F_2(da)$ is analogous to the Steinberg relation in Milnor K-theory. (For $a \in A$ this corresponds to

$$d\log[a] \cdot d\log[a] = d\log[-1]d\log[a]$$

which we compare to the relation $\{a, a\} = \{-1, a\}$ in Milnor K-theory.)

We will mention some of the important properties of this construct (and show some of them).

Lemma 5.9. *The graded $\mathbb{W}(A)$-algebra $\widehat{\Omega}_{\mathbb{W}(A)}$ is anticommutative.*

Proof. We have to show that for $a, b \in \mathbb{W}(A)$ the sum $da \cdot db + db \cdot da \in \widehat{\Omega}^2_{\mathbb{W}(A)}$ equals zero. we compute first using the defining relations in two ways:

$$d(a+b) \cdot d(a+b) = d\log[-1] \cdot F_2 d(a+b) = d\log[-1] \cdot F_2 da + d\log[-1] \cdot F_2 db$$

and

$$d(a+b) \cdot d(a+b) = da \cdot da + da \cdot db + db \cdot da + db \cdot db = d\log[-1] \cdot F_2 da + da \cdot db + db \cdot da + d\log[-1] \cdot F_2 db$$

Comparing the two expressions shows that $db \cdot da = da \cdot db$. ☐

Proposition 5.10. *There exists a unique graded derivation*

$$d : \widehat{\Omega}_{\mathbb{W}(A)} \to \widehat{\Omega}_{\mathbb{W}(A)}$$

extending the derivation $d : \mathbb{W}(A) \to \Omega^1_{\mathbb{W}(A)}$ and satisfying

$$dd\omega = d\log[-1] \cdot d\omega.$$

Moreover, the element $d\log[-1]$ is a cycle.

Proof. Inductively, the map d will be given for $a_0, \dots, a_q \in \mathbb{W}(A)$

$$d(a_0 da_1 \cdots da_q) = da_0 \cdots da_q + qd\log[-1] \cdot a_0 da_1 \cdots da_q$$

whoch means that the second summand disappears for q even and equals $d\log[-1] \cdot a_0 da_1 \cdots da_q$ for q odd. If the so defined map is a well defined graded derivation satisfying the relation $dd\omega = d\log[-1] \cdot d\omega$, it is necessarily unique. This is left to the reader as exercise.

It then follows from $dd\omega = d\log[-1] \cdot d\omega$ that $d\log[-1]$ is in fact a cycle:

$$\begin{aligned} d(d\log[-1]) &= d([-1]d[-1]) \\ &= d[-1] \cdot d[-1] + [-1]dd[-1] \\ &= d\log[-1] \cdot F_1 d[-1] + [-1]d\log[-1]d[-1] \\ &= d\log[-1] \cdot [-1]d\log[-1] + [-1]d\log[-1]d[-1] \\ &= 2(d\log[-1] \cdot [-1]d[-1]) = 0 \end{aligned}$$

(because $\widehat{\Omega}_{\mathbb{W}(A)}$ is anticommutative). □

Note that in general there is no $\mathbb{W}(A)$-algebra map $\widehat{\Omega}_{\mathbb{W}(A)} \to \Omega_{\mathbb{W}(A)}$ compatible with the derivations!

Proposition 5.11. *Let A be a ring and $n \in \mathbb{N}$. There is a unique homomorphism of graded rings*

$$F_n : \widehat{\Omega}_{\mathbb{W}(A)} \to \widehat{\Omega}_{\mathbb{W}(A)}$$

extending F_n from degree 0 and 1. Additionally

$$dF_n = nF_n d.$$

Proof. Similar to th definition of d, the map F_n has to be given by

$$F_n(a_0 da_1 \cdots da_q) = F_n(a_0)F_n(da_1) \dots F_n(da_q)$$

to be a graded ring homomorphism extending F_n from degrees 0 and 1, and this is unique if it is well defined. To show this, one has to sow that

$$F_n(da)F_n(da) = F_n(d\log[-1])F_n(F_2 da)$$

It suffices to show this for $n = p$ prime. This is left to the reader.

The formula $dF_n = nF_n d$ is already known in degree 1. Again, wlog, we can assume $n = p$ to be prime. To extend this to higher degrees, let $a \in \mathbb{W}(A)$. Then

$$dF_p(da) = d(a^{p-1}da + d\left(\frac{F_p(a) - a^p}{p}\right) = (p-1)a^{p-2}dada + d\log[-1] \cdot F_p da$$

which is 0 for $p = 2$ by the defining relations, and equal to $d\log[-1] \cdot F_p da$ of p is odd (because then $p-1$ is even which kills the first summand). Induction give the formula for higher degrees than 2. □

So far, we hae established some important additional structures on $\widehat{\Omega}_{\mathbb{W}(A)}$ however, Verschiebung does in general not extend to this $\mathbb{W}(A)$ algebra. We therefore define a quotient of it, where in degree 1 the desired relation between Verschiebung, Frobenius and derivation holds.

Definition 5.12. Let A be a ring. Set

$$\check{\Omega}_{\mathbb{W}(A)} = \widehat{\Omega}_{\mathbb{W}(A)}/K$$

where K is the graded ideal generated by the elements

$$F_p dV_p(a) - da - (p-1)d\log[-1] \cdot a$$

for all primes p and all $a \in \mathbb{W}(A)$. This is a graded $\mathbb{W}(A)$-algebra.

Note that the element $F_p dV_p(a) - da - (p-1)d\log[-1]\cdot a$ is annihilated by p (in particular, it is zero if p is invertible in A and hence in $\mathbb{W}(A)$).

In order for this definition to be useful, the maps F_n and d should descent from $\widehat{\Omega}_{\mathbb{W}(A)}$.

Lemma 5.13. *For all $n \in \mathbb{N}$ the Frobenius map $F_n : \widehat{\Omega}_{\mathbb{W}(A)} \to \widehat{\Omega}_{\mathbb{W}(A)}$ induces a map of graded algebras*

$$F_n : \check{\Omega}_{\mathbb{W}(A)} \to \check{\Omega}_{\mathbb{W}(A)}.$$

The graded derivation $d : \widehat{\Omega}_{\mathbb{W}(A)} \to \widehat{\Omega}_{\mathbb{W}(A)}$ induces a graded derivation

$$d : \check{\Omega}_{\mathbb{W}(A)} \to \check{\Omega}_{\mathbb{W}(A)}.$$

Moreover, or all $n \in \mathbb{N}$ and $a \in \mathbb{W}(A)$

$$F_n dV_n(a) = da + (n-1)d\log[-1]\cdot a$$

holds in $\check{\Omega}^1_{\mathbb{W}(A)}$.

Proof. The calculations to do here are not difficult, and in general obvious, but a bit tedious. □

So far, the definitions hold for the big Witt vectors, meaning that $S = \mathbb{N}$. But using restriction, the other cases are covered as well.

Definition 5.14. Let A be a ring, $S \subset \mathbb{N}$ a truncation set and $I_S(A) \subset \mathbb{W}(A)$ the kernel of $R^{\mathbb{N}}_S : \mathbb{W}(A) \to \mathbb{W}_S(A)$. The maps

$$\widehat{\Omega}_{\mathbb{W}(A)} \xrightarrow{R^{\mathbb{N}}_S} \widehat{\Omega}_{\mathbb{W}_S(A)} \quad \text{and} \quad \check{\Omega}_{\mathbb{W}(A)} \xrightarrow{R^{\mathbb{N}}_S} \check{\Omega}_{\mathbb{W}_S(A)}$$

are the quotient maps that annihilate the respective graded ideals generated by $I_S(A)$ and $dI_S(A)$.

Lemma 5.15. *The derivation, restriction and Frobenius defined before induce maps*

$$\begin{array}{ll} d : \widehat{\Omega}_{\mathbb{W}_S(A)} \to \widehat{\Omega}_{\mathbb{W}_S(A)} & d : \check{\Omega}_{\mathbb{W}_S(A)} \to \check{\Omega}_{\mathbb{W}_S(A)} \\ R^T_S : \widehat{\Omega}_{\mathbb{W}_S(A)} \to \widehat{\Omega}_{\mathbb{W}_T(A)} & R^T_S : \check{\Omega}_{\mathbb{W}_S(A)} \to \check{\Omega}_{\mathbb{W}_T(A)} \\ F_n : \widehat{\Omega}_{\mathbb{W}_S(A)} \to \widehat{\Omega}_{\mathbb{W}_{\frac{S}{n}}(A)} & F_n : \check{\Omega}_{\mathbb{W}_S(A)} \to \check{\Omega}_{\mathbb{W}_{\frac{S}{n}}(A)} \end{array}$$

The maps d are graded derivations, the maps R^T_S and F_n are graded ring homomorphisms; R^T_S and d commute and $dF_n = nF_n d$.

Proof. For the first part, there are a few equations to check. The second part is clear. □

Now we want to extend the commuting diagram for a Witt complex E_S

$$\begin{tikzcd} \Omega^1_{\mathbb{W}_S(A)} \arrow[r, "\eta_S"] \arrow[d, "F_n"] & E^1_S \arrow[d, "F_n"] \\ \Omega^1_{\mathbb{W}_{\frac{S}{n}}(A)} \arrow[r, "\eta_{\frac{S}{n}}"] & E^1_{\frac{S}{n}} \end{tikzcd}$$

to $\check{\Omega}_{\mathbb{W}_S(A)}$.

Proposition 5.16. *Let E_S be a Witt complex over the ring A. There is a unique natural homomorphism of graded rings*

$$\eta_S : \check{\Omega}_{\mathbb{W}_S(A)} \to E_S$$

that extends the natural ring homomorphism $\eta_S : \mathbb{W}_S(A) \to E^0_S$ and commutes with derivations. For $m \in \mathbb{N}$ the diagram

$$\begin{tikzcd} \check{\Omega}_{\mathbb{W}_S(A)} \arrow[r, "\eta_S"] \arrow[d, "F_m"] & E_S \arrow[d, "F_m"] \\ \check{\Omega}_{\mathbb{W}_{\frac{S}{m}}(A)} \arrow[r, "\eta_{\frac{S}{m}}"] & E_{\frac{S}{m}} \end{tikzcd}$$

commutes.

Proof. As before, there is no other way the map η_S can be given than by

$$\eta_S(a_0 da_1 \cdots da_q) = \eta_S(a_0) d\eta_S(a_1) \cdots d\eta_S(a_q)$$

To show that it is well defined, we note first from the proposition in degree 1 that

$$F_2 d\eta_{\mathbb{N}}(a) = \eta_{\mathbb{N}} F_2 d(a) = \eta_{\mathbb{N}} \left(ada + d\left(\frac{F_2(a) - a^2}{2} \right) \right) = \eta_{\mathbb{N}}(a) d\eta_{\mathbb{N}}(a) + d\eta_{\mathbb{N}} \left(\frac{F_2(a) - a^2}{2} \right)$$

Now we apply d to this equation, so that the left hand side becomes

$$dF_2 d\eta_{\mathbb{N}}(a) = 2F_2 dd\eta_{\mathbb{N}}(a) = 0$$

and the right hand side reads

$$d\eta_{\mathbb{N}}(a) d\eta_{\mathbb{N}}(a) + d\log \eta_{\mathbb{N}}([-1]_{\mathbb{N}}) \cdot (\eta_{\mathbb{N}}(a) d\eta_{\mathbb{N}}(a) + d\eta_{\mathbb{N}} \left(\frac{F_2(a) - a^2}{2} \right)) = d\eta_{\mathbb{N}}(a) d\eta_{\mathbb{N}}(a) d\log \eta_{\mathbb{N}}([-1]_{\mathbb{N}}) \cdot F_2 d\eta_{\mathbb{N}}(a)$$

and together the equation

$$0 = d\eta_{\mathbb{N}}(a) d\eta_{\mathbb{N}}(a) d\log \eta_{\mathbb{N}}([-1]_{\mathbb{N}}) \cdot F_2 d\eta_{\mathbb{N}}(a)$$

which is the defining relation of $\widehat{\Omega}_{\mathbb{W}_S(A)}$. Thus the above defined map is well defined on $\widehat{\Omega}_{\mathbb{W}_S(A)} \to E_S$. Moreover this map factors through $\check{\Omega}_{\mathbb{W}_S(A)}$ which is the quotient of $\widehat{\Omega}_{\mathbb{W}_S(A)}$ by the ideal generated by $F_p dV_p(a) - da - (p-1) d\log[-1] \cdot a$ because o point (4) of the definition of Witt complexes. Finally it is clear from the definition of η_S above, and from the equivalent result in degree 1, that the desired diagram commutes. □

The existence of the Forbenius used here follows quite explicitely from the theory λ-rings, and modules and derivations over those, which will be the subject of the following section.

5.3 Modules and derivations over λ-rings

We already mentioned the following fact, when we introduced the big Witt vectors. For simplicity, denote $\mathbb{W}(A) := \mathbb{W}_{\mathbb{N}}(A)$ for a ring A as above.

Proposition 5.17. *There exists a unique natural ring homomorphism*

$$\Delta = \Delta_A : \mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(A))$$

such that for any $n \in \mathbb{N}$

$$w_n \circ \Delta = F_n : \mathbb{W}(A) \to \mathbb{W}(A).$$

In addition, the following diagrams, with $\varepsilon_B = w_1 : \mathbb{W}(B) \to B$ for a ring B, commute

$$\begin{tikzcd} \mathbb{W}(A) & \mathbb{W}(\mathbb{W}(A)) \arrow[l, "\varepsilon_{\mathbb{W}(A)}"'] \arrow[r, "\mathbb{W}(\varepsilon_A)"] & \mathbb{W}(A) \\ & \mathbb{W}(A) \arrow[u, "\Delta_A"'] \arrow[ul, equal] \arrow[ur, equal] & \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{W}(\mathbb{W}(\mathbb{W}(A))) & \mathbb{W}(\mathbb{W}(A)) \arrow[l, "\Delta_{\mathbb{W}(A)}"'] \\ \mathbb{W}(\mathbb{W}(A)) \arrow[u, "\mathbb{W}(\Delta_A)"] & \mathbb{W}(A) \arrow[l, "\Delta_A"'] \arrow[u, "\Delta_A"'] \end{tikzcd}$$

Proof. To prove existence, it is enough to do that in the universal case $A = \mathbb{Z}[a_1, a_2, \ldots]$ and $a = (a_1, a_2, \ldots)$ there is an element $\Delta_A(a) \in \mathbb{W}(\mathbb{W}(A))$ with image under the ghost map

$$w : \mathbb{W}(\mathbb{W}(A)) \to \mathbb{W}(A)^{\mathbb{N}}$$

is $(F_n(a))_{n\in\mathbb{N}}$. Since w in this universal case is injective, the element $\Delta_A(a)$ is unique - if it exists.

By Dworks Lemma and the definition of F_p, $(F_n(a))$ is in the image of the ghost map, iff for $p \in \mathbb{N}$ prime and $n \in p\,\mathbb{N}$

$$F_n(a) \equiv F_p(F_{\frac{n}{p}}) \bmod p^{\nu_p(n)}\, \mathbb{W}(A),$$

which follows from $F_n([a]_S) = [a]_{\frac{S}{n}}^{n}$.

Thus existence and uniqueness of the map Δ. One checks that the diagrams commute by computing them in ghost coordinates. □

Note that the map $\Delta_n : \mathbb{W}(A) \to \mathbb{W}(A)$ given by the n$^{\text{th}}$ component of Δ is in general not a ring homomorphism.

Moreover, for $a \in A$: $\Delta([a]) = [[a]]$.

This natural transformation is called the universal λ-operation. With this, Grothendieck's definition of λ-rings can be stated as follows.

Definition 5.18. A λ-ring is a pair (A, λ), where A is a ring, and $\lambda : A \to \mathbb{W}(A)$ such that the diagrams

$$\begin{tikzcd} A & \mathbb{W}(A) \arrow[l, "\varepsilon_A"'] \\ & A \arrow[u, "\lambda"'] \arrow[ul, equal] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathbb{W}(\mathbb{W}(A)) & \mathbb{W}(A) \arrow[l, "\Delta_A"'] \\ \mathbb{W}(A) \arrow[u, "\mathbb{W}(\lambda)"] & A \arrow[l, "\lambda"'] \arrow[u, "\lambda"'] \end{tikzcd}$$

commute. A morphism of λ-rings $f : (A, \lambda_A) \to (B, \lambda_B)$ is a ring homomorphism $f : A \to B$ such that

$$\lambda_B \circ f = \mathbb{W}(f) \circ \lambda_A.$$

For a λ-ring (A, λ) we denote by $\lambda_n : A \to A$ the n^{th} Witt component of $\lambda(a)$. The so defined map is in general neither additive nor multiplicative.

Definition 5.19. Let (A, λ) be a λ-ring. The associated n^{th} Adams operation is the composite ring homomorphisms

$$\psi_n = w_n \circ \lambda : A \to A.$$

We mention some results:

Lemma 5.20. *Let (A, λ) be a λ-ring. The associated Adams operations satisfy:*

1. *the map $\psi_1 = \mathrm{id}_A$*
2. *for all positive integers $m, n \in \mathbb{N}$: $\psi_m \circ \psi_n = \psi_{mn}$*
3. *for a prime $p \in \mathbb{N}$, $a \in A$: $\psi_p(a) \equiv a^p \bmod pA$*

Proof. The properties (1) and (3) follow directly from the definition. (2) follows from

$$\begin{aligned} \psi_m \circ \psi_n &= w_m \circ \lambda \circ w_n \circ \lambda \\ &= w_m \circ w_n \circ \mathbb{W}(\lambda) \circ \lambda \quad \text{from naturality of } w_n \\ &= w_m \circ w_n \circ \Delta \circ \lambda \quad \text{by definition of a } \lambda\text{-ring} \\ &= W_m \circ F_n \circ \lambda \quad \text{by definition of } \Delta \\ &= w_{mn} \circ \lambda = \psi_{mn} \quad \text{by definition of } F_n \end{aligned}$$

□

Proposition 5.21 (Wilkerson)**.** *If A is a flat ring over $\mathbb{Z}$, with a family of ring endomorphisms ψ_n satisfying properties (1)-(3) from the previous lemma. Then there is a unique λ-ring structure on A for which the ψ_n are the associated Adams operations.*

Proof. This can be found in [14]. □

Lastly, we cite a result obtained independently by Borger [3, 4] and van der Kallen [13].

Theorem 5.22. *Let $f : A \to B$ be étale, S a finite truncation set, $n \in \mathbb{N}$. Then the induced morphism*

$$\mathbb{W}_S(f) : \mathbb{W}_S(A) \to \mathbb{W}_S(B)$$

is étale and the diagram

$$\begin{tikzcd} \mathbb{W}_S(A) \arrow[r, "\mathbb{W}_S(f)"] \arrow[d, "F_n"] & \mathbb{W}_S(B) \arrow[d, "F_n"] \\ \mathbb{W}_{\frac{S}{n}}(A) \arrow[r, "\mathbb{W}_{\frac{S}{n}}(f)"] & \mathbb{W}_{\frac{S}{n}}(B) \end{tikzcd}$$

is cocartesian.

The definition of modules over λ-rings used by Hesselholt in [7, Sec. 2] is based on the following definition employed by Beck [1] in his thesis.

Let $\mathcal{C}$ be a category with finite limits and $X \in \mathcal{C}$. Then the category of X-modules $(\mathcal{C}/X)_{\mathrm{ab}}$ is the category of abelian group objects in $\mathcal{C}$ over X. The derivations from X to the X-module $(Y/X, +_Y, 0_Y, -_Y)$ is the set

$$\mathrm{Der}(X, (Y/X, +_Y, 0_Y, -_Y)) = \mathrm{Hom}_{\mathcal{C}/X}(X/X, Y/X).$$

We will use this as a working definition.

Remark 5.23. A few reminders about category theory.

In general an adjunction from a category $\mathscr{C}$ to a category $\mathscr{D}$ is a quadruple $(F, G, \varepsilon, \eta)$ where $F : \mathscr{C} \to \mathscr{D}$ and $G : \mathscr{D} \to \mathscr{C}$ are functors, and $\varepsilon : F \circ G \Rightarrow \mathrm{id}$ and $\eta : G \circ F \Rightarrow \mathrm{id}$ are natural transformations, such that

$$F \xRightarrow{F\circ\eta} F \circ G \circ F \xRightarrow{\varepsilon\circ F} F \qquad \text{and} \qquad G \xRightarrow{\eta\circ G} G \circ F \circ G \xRightarrow{G\circ\varepsilon} G$$

are equal to the respective identity natural transformation. This is often refer to as triangle identities. The transformations ε and η are called counit and unit of the adjunction. The adjunction is calle adjoint equivalence, if they are both isomorphisms.

A functor $G : \mathscr{D} \to \mathscr{C}$ admits a left adjoint if an adjunction $(F, G, \varepsilon, \eta)$ exists. F is then called a left adjoint of R. If a left adjoint exists, then it is unique up to unique isomorphism. Similar for right adjoints.

Let $\mathscr{A}$ be the category of (commutative) rings. For $A \in \mathscr{A}$ we define an adjunction $(F, G, \varepsilon, \eta)$ from the category $(\mathscr{A}/A)_{\mathrm{ab}}$ of A-modules as defined above (abelian group objects in the category $\mathscr{A}/A$), to the category $\mathscr{M}(A)$ of A-modules in the usual sense:

Let $f : B \to A$ be in $\mathscr{A}/A$ and the abelian group structure given by

$$\begin{tikzcd} B \times_f B \arrow[r, "+_B"] \arrow[d] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "0_B"] \arrow[d, equal] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "-_B"] \arrow[d, "f"] & B \arrow[d, "f"] \\ A \arrow[r, equal] & A \end{tikzcd}$$

Then F associates to the abelian group object $(f, +_B, 0_B, -_B)$ the A-module $M = \mathrm{Ker}(f)$ with the A-module structure

$$a.x = 0_B(a)x.$$

On the other hand, if M is an A-module, let $A \ltimes M$ be the ring given by $A \oplus M$ with multiplication

$$(a, x).(a', x') = (aa', ax' + a'x)$$

and let $G(M)$ be the group object $(f, +, 0, -)$ with $f : A \ltimes M \to A$ the projection, $(a, x) + (a, x') = (a, x + x')$, $0(a) = (a, 0)$ and $-(a, x) = (a, -x)$. We define $\varepsilon : G \circ F \Rightarrow \mathrm{id}$ and $\eta : F \circ G \Rightarrow \mathrm{id}$ by

$$\varepsilon(a, x) = 0_B(a) + x \qquad \text{and} \qquad \eta(x) = (0, x)$$

Lemma 5.24. *If A is a ring, then the quadruple $(F, G, \eta, \varepsilon)$ is an adjoint equivalence of categories from $(\mathscr{A}/A)_{ab}$ to $\mathscr{M}(A)$.*

Proof. This is a result due to Beck and will be done in the exercise session. □

We will look at the analogous statement for λ-rings.

Before, we will study the Witt vectors of the ring $A \ltimes M$ defined earlier. Recall that the polynomials $s_n(\underline{a}, \underline{b}), p_n(\underline{a}, \underline{b}), i_n(\underline{a})$ which define the sum product and inverse in the ring of (big) Witt vectors have constant term 0. Thus the (big) Witt vectors can be defined for non-unital rings as well. Moreover, by induction one sees that they are congruent to

$$\begin{aligned} s_n(\underline{a}, \underline{b}) &\equiv a_n + b_n \\ p_n(\underline{a}, \underline{b}) &\equiv a_n b_n \\ i_n(\underline{a}) &\equiv -a_n \end{aligned}$$

modulo higher degrees. If we consider the module M as non-unital ring with zero multiplication, then its Witt ring $\mathbb{W}_S(M)$ has also zero multiplication, and has underlying additive group M^S with componentwise addition.

Similarly, one shows, that the polynomials defining the Frobenius and the universal λ-operation have constant term zero and are congruent to na_{nm} for F_n and a_{nm} for Δ_n resepectively, so that

$$\begin{aligned} F_n : &\quad \mathbb{W}_S(M) \to \mathbb{W}_{\frac{S}{n}}(M),\ (x_m)_{m\in S} \mapsto (nx_{nm})_{m\in\frac{S}{n}} \\ \Delta_M : &\quad \mathbb{W}(M) \to \mathbb{W}(\mathbb{W}(M)),\ (x_m)_{m\in\mathbb{N}} \mapsto ((x_{me})_{e\in\mathbb{N}})_{m\in\mathbb{N}} \end{aligned}$$

Lemma 5.25. *Let S be a truncation set, A a ring and M an A-module. Assume that $\mathbb{W}_S(M)$ is endowed witht the $\mathbb{W}_S(A)$-module structure such that for $a \in \mathbb{W}_S(A)$ and $x \in \mathbb{W}_S(M)$, $ax \in \mathbb{W}_S(M)$ has Witt components $(ax)_n = w_n(a)x_n$. Then the canonical inclusions $i_1 : A \to A \ltimes M$ and $i_2 : M \to A \ltimes M$ induce a ring isomorphism*

$$i_{1*} + i_{2*} : \mathbb{W}_S(A) \ltimes \mathbb{W}_S(M) \to \mathbb{W}_S(A \ltimes M).$$

Proof. Consider the diagram of rings

$$0 \longrightarrow M \xrightarrow{i_2} A \ltimes M \underset{i_1}{\overset{p_1}{\rightleftarrows}} A \longrightarrow 0$$

Although not a priori exact as diagram of rings, it is split exact seen as diagram of additive groups. Likewise, the induced diagram of rings

$$0 \longrightarrow \mathbb{W}_S(M) \xrightarrow{i_{2*}} \mathbb{W}_S(A \ltimes M) \underset{i_{1*}}{\overset{p_{1*}}{\rightleftarrows}} \mathbb{W}_S(A) \longrightarrow 0$$

has an underlying diagram of additive groups which is split exact. It follows that the map of the statement is an isomorphism of additive groups. Moreover, it is a morphism of rings, if $\mathbb{W}_S(M)$ is given the $\mathbb{W}_S(A)$-module structure such that $i_{2*}(ax) = i_{1*}(a)i_{2*}(x)$ for all $a \in \mathbb{W}_S(A)$ and $x \in \mathbb{W}_S(M)$. It remains to show that ax equals the Witt vector y with components $w_n(a)x_n$. Wlog, we may assume that A and M are torsion free (otherwise, we can find a surjection from a torsion free ring). In this case, the ghost map is injective, so that we can use ghost components to show the claim. In other words, for each $n \in \mathbb{N}$ we have to show $w_n(ax) = w_n(y)$ in $\mathbb{W}_S(M)$, which means we have to show $i_2(w_n(ax)) = i_2(w_n(y))$ in $\mathbb{W}_S(A \ltimes M)$. Bearing in mind that w_n is a ring homomorphism we compute

$$\begin{aligned} i_2(w_n(ax)) &= w_n(i_{2*}(ax)) \\ &= w_n(i_{1*}(a)i_{2*}(x)) \\ &= w_n(i_{1*}(a))w_n(i_{2*}(x)) \\ &= i_1(w_n(a))i_2(w_n(x)) \\ &= i_2(w_n(a)w_n(x)) \\ &= i_2(nw_n(a)x_n) \\ &= i_2(ny_n) = i_2(w_n(y)) \end{aligned}$$

which proves the claim. □

To describe the elements of $\mathbb{W}_S(A \ltimes M)$ we prove the following:

Lemma 5.26. *Let A, M, S be as above, $a \in \mathbb{W}_S(A)$ and $x \in \mathbb{W}_S(M)$. Then the Witt components $b_n = a_n.y_n \in A \ltimes M$ of $b = i_{1*}(a) + i_{2*}(x) \in \mathbb{W}_S(A \ltimes M)$ satisfy*

$$\sum_{e|n} a_e^{\frac{n}{e}-1} y_e = x_n.$$

Proof. This is an exercise. □

Inspired by this, we now consider for a ring A and an A-module M and truncation set S the $\mathbb{W}_S(A)$-module $\mathbb{W}_S(M)$ to be the set M^S with component wise addition and with scalar multiplication defined for $a \in \mathbb{W}_S(A), x \in \mathbb{W}_S(M)$ by

$$(ax)_n = \psi_{A,n}(a)x_n$$

where $\psi_{A,n}$ is the n^{th} Adams operation of A.

Remark 5.27. In the case, when M is the A-module A itself, then the $\mathbb{W}_S(A)$-modules $\mathbb{W}_S(M)$ defined as above is in general not the same as the $\mathbb{W}_S(A)$-module $\mathbb{W}_S(A)$ via multiplication.

Now back to our goal to prove a λ-ring equivalent of Lemma 5.24. For this, we first give a straight forward definition of modules in this context.

Definition 5.28. Let (A, λ_A) be a λ-ring. An (A, λ_A)-module is a pair $(M, \lambda_M)$where M is an A-module and

$$\lambda_M :\to \mathbb{W}(M)$$

a λ_A-linear map such that the diagrams

$$\begin{tikzcd} M & \mathbb{W}(M) \arrow[l, "\varepsilon_M"'] \\ & M \arrow[ul, equal] \arrow[u, "\lambda_M"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathbb{W}(\mathbb{W}(M)) & \mathbb{W}(M) \arrow[l, "\Delta_M"'] \\ \mathbb{W}(M) \arrow[u, "\mathbb{W}(\lambda_M)"'] & M \arrow[l, "\lambda_M"'] \arrow[u, "\lambda_M"'] \end{tikzcd}$$

commute.

A morphism $h : (M, \lambda_M) \to (N, \lambda_N)$ of (A, λ_A)-modules is an A-linear map $h : M \to N$ such that

$$\lambda_N \circ h = \mathbb{W}(h) \circ \lambda_M.$$

Denote by $\mathscr{M}(A, \lambda_A)$ the category of (A, λ_A)-modules.

Example 5.29. For a λ-ring (A, λ_A) one can define an (A, λ_A)-module by setting $(M, \lambda_M) = (A, \psi_A)$. Note however, that (A, λ_A) itself is in general not an (A, λ_A)-module.

As we have seen for a ring A $(\mathbb{W}(A), \Delta_A)$ is a λ-ring. In fact, the functor, $R : A \mapsto (\mathbb{W}(A), \Delta_A)$ is right adjoint to the forgetful functor

$$U : \mathscr{A}_\lambda \to \mathscr{A}$$

(with unit given by $\lambda : (A, \lambda_A) \to (\mathbb{W}(A), \Delta_A)$ and counit by $\varepsilon_A : \mathbb{W}(A) \to A$).

We also have an adjunction

$$\begin{tikzcd} \mathscr{A}_\lambda/(A, \lambda_A) \arrow[r, shift left, "U_{(A,\lambda_A)}"] & \mathscr{A}/A \arrow[l, shift left, "R_{(A,\lambda_A)}"] \end{tikzcd}$$

where the forgetfulfunctor $U_{(A,\lambda_A)}$ takes $f : (B, \lambda_B) \to (A, \lambda_A)$ to $f : B \to A$ and its right adjoint takes $f : B \to A$ to the pullback $p_2 : (C, \lambda_C) \to (A, \lambda_A)$ with

$$\begin{tikzcd} (C, \lambda_C) \arrow[r, "p_1"] \arrow[d, "p_2"'] & (\mathbb{W}(B), \Delta_B) \arrow[d, "\mathbb{W}(f)"] \\ (A, \lambda_A) \arrow[r, "\lambda_A"'] & (\mathbb{W}(A), \Delta_A) \end{tikzcd}$$

Since both functors preserve limits, as the functors above, they induce an adjunction on the subcategory of abelian group objects

$$(\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}} \rightleftarrows (\mathscr{A}/A)_{\mathrm{ab}}$$

which correspond to the adjunction

$$\begin{tikzcd} \mathscr{M}(A,\lambda_A) \arrow[r, shift left, "U'"] & \mathscr{M}(A) \arrow[l, shift left, "R'"] \end{tikzcd}$$

$$(M,\lambda_M) \longmapsto M$$

$$(\lambda_{A*}(\mathbb{W}(N)),\Delta_N) \longleftarrow\!\shortmid N$$

The notation $\lambda_{A*}(\mathbb{W}(N))$ means the $\mathbb{W}(A)$-modules $\mathbb{W}(N)$ considered as an A module via λ_A.

We now come to the analogue of Beck's result.

Proposition 5.30. *Let (A,λ_A) be a λ-ring. There exist a unique adjunction (up to unique isomorphism)*

$$(F^\lambda, G^\lambda, \varepsilon^\lambda, \eta^\lambda) : (\mathscr{A}_\lambda/(A,\lambda_A))_{ab} \to \mathscr{M}(A,\lambda_A)$$

such that in the diagram below the square of left adjoint functors commutes

$$\begin{tikzcd} (\mathscr{A}/A)_{ab} \arrow[r, shift left, "F"] \arrow[d, shift left, "R_{(A,\lambda_A)}"] & \mathscr{M}(A) \arrow[l, shift left, "G"] \arrow[d, shift left, "R'"] \\ (\mathscr{A}_\lambda/(A,\lambda_A))_{ab} \arrow[u, shift left, "U_{(A,\lambda_A)}"] \arrow[r, shift left, "F^\lambda"] & \mathscr{M}(A,\lambda_A) \arrow[l, shift left, "G^\lambda"] \arrow[u, shift left, "U'"] \end{tikzcd}$$

Moreover, this defines an equivalence of categories.

Proof. Recall that F was defined by associating to an abelian group object $(f : B \to A, +_B, 0_b, -_B)$ the A-module $M = \ker f$ with the module structure $a\dot{y} = 0_B(a)x$. And G was defined by sending an A-module M to the group object $(f : A \ltimes M \to A, +, 0, -)$.

Now let $(f : (B,\lambda_B) \to (A,\lambda_A), +_B, 0_B, -_B) \in (\mathscr{A}_\lambda/(A,\lambda_A))_{\mathrm{ab}}$, then $F^\lambda(f, +_B, 0_B, -_B) = (M,\lambda_M)$ with $M = F(f)$ and $\lambda_M : M \to \mathbb{W}(M)$ induced by functoriality on the kernels of the vertical maps in

$$\begin{tikzcd} B \arrow[r, "\lambda_B"] \arrow[d, "f"'] & \mathbb{W}(B) \arrow[d, "\mathbb{W}(f)"'] \\ A \arrow[r, "\lambda_A"] & \mathbb{W}(A) \end{tikzcd}$$

and it is clear that $U' \circ F^\lambda = F \circ U_{(A,\lambda_A)}$.

Conversely, for an (A,λ_A)-module (M,λ_M), let $G^\lambda(M,\lambda_M)$ be $G(M)$ of above (with underlying ring $B = A \ltimes M$), with the *lambda*-ring structure $\lambda_B : B \to \mathbb{W}(B)$ given by

$$A \ltimes M \xrightarrow{\lambda_A \oplus \lambda_M} \mathbb{W}(A) \ltimes \mathbb{W}(M) \xrightarrow{i_1+i_2} \mathbb{W}(A \ltimes M)$$

One then has to show that G^λ is well-defined, for which one needs the three following steps:

1. (B,λ_B) is a λ-ring.
2. The canonical projection $f : (B,\lambda_B) \to (A,\lambda_A)$ is a λ-ring morphism.
3. The abelian group object structure maps $+_B$, 0_B and $-B$ on $f : B \to A$ are λ-ring morphisms.

The proof of these tree statements involve the techniques that we discussed earlier on Witt vectors of modules. The reader is encouraged to do this. Note also, that by construction

$$U_{(A,\lambda_A)} \circ G^\lambda = G \circ U'.$$

Lastly, one has to show that F^λ and G^λ form an adjoint pair compatible with the adjoint pair (F, G), meaning there are unique natural isomorphisms (transformations)

$$G^\lambda \circ F^\lambda \xRightarrow{\varepsilon^\lambda} \mathrm{id} \qquad \text{and} \qquad \mathrm{id} \xRightarrow{\eta^\lambda} F^\lambda \circ G^\lambda$$

such that

$$U_{(A,\lambda_A)}(\varepsilon^\lambda) = \varepsilon \circ U_{(A,\lambda_A)} \text{ and } U'(\eta^\lambda) = \eta \circ U'$$

This means commutativity of the following two diagrams where M is a λ-module, B is the λ-ring $A \ltimes M$ as above, $i : M \to B$ is a chosen embedding of the kernel of $f : B \to A$ into B, of which the first one corresponds to G^λ and the second one corresponds to F^λ.

$$\begin{tikzcd}
M \arrow[r, "\lambda_M"] \arrow[d, "i_2"] & \mathbb{W}(M) \arrow[r, equal] \arrow[d, "i_2"] & \mathbb{W}(M) \arrow[d, "i_{2*}"] \\
A \ltimes M \arrow[r, "\lambda_A \oplus \lambda_M"] & \mathbb{W}(A) \ltimes \mathbb{W}(M) \arrow[r, "i_{1*}+i_{2*}"] & \mathbb{W}(A\ltimes)
\end{tikzcd}$$

$$\begin{tikzcd}
A \ltimes M \arrow[r, "\lambda_A \oplus \lambda_M"] \arrow[d, "0_B+i"] & \mathbb{W}(A) \ltimes \mathbb{W}(M) \arrow[r, "i_{1*}+i_{2*}"] \arrow[d, "0_{B*}+i_*"] & \mathbb{W}(A\ltimes) \arrow[d, "(0_B+i)_*"] \\
B \arrow[r, "\lambda_B"] & \mathbb{W}(B) \arrow[r, equal] & \mathbb{W}(B)
\end{tikzcd}$$

In both diagrams, the left-hand squares commute by naturality and the right-hand squares by the universal property of the direct sum. $\square$

It will be advantageous to be able to work in either category.

We will now define derivations on $\mathscr{M}(A, \lambda_A)$ and bring them together with Beck's more general definition.

Definition 5.31. Let (A, λ_A) be a λ-ring, and (M, λ_M) an (A, λ_A)-module. A derivation

$$D : (A, \lambda_A) \to (M, \lambda_M)$$

is a map of sets such that

1. **Additivity**: for $a, b \in A$, $D(a + b) = D(a) + D(b)$
2. **Leibniz rule**: for $a, b \in A$, $D(ab) = aD(b) + bD(a)$
3. **λ-semilinearity**: for $a \in A$ and $n \in \mathbb{N}$, $\lambda_{M,n}(D(a)) = \sum_{e|n} \lambda_{A,e}(a)^{\frac{n}{e}-1} D(\lambda_{A,e}(a))$

The set of derivations is denoted by $\mathrm{Der}((A, \lambda_A), (M, \lambda_M))$.

Under the equivalence of Prop. 5.30 we have:

Proposition 5.32. *Let (A, λ_A) be a λ-ring, (M, λ_M) and (A, λ_A)-module, and $f : (A \ltimes M, \lambda_{A \ltimes M}) \to (A, \lambda_A)$ the canonical projection. Then there is a bijection*

$$\begin{array}{rcl} \mathrm{Der}((A, \lambda_A), (M, \lambda_M)) & \to & \mathrm{Hom}_{\mathscr{A}_\lambda/(A,\lambda_A)}(\mathrm{id}_{(A,\lambda_A)}, f) \\ D & \mapsto & (\mathrm{id}_A, D) \end{array}$$

Proof. Without λ it is easily verified, that the map from $\mathrm{Der}(A, M)$ to $\mathrm{Hom}_{\mathscr{A}/A}(\mathrm{id}_A, f)$ taking D to (id_A, D) is a bijection.

By abuse of notation, we also write $(\mathrm{id}_A, D) : A \to A \ltimes M$ without the underlying maps. In order to show the claim, we have to show that D is a λ-derivation – meaning, we have to check λ-linearity – iff $(\mathrm{id}_A, D) : A \to A \ltimes M$ is a λ-ring homomorphism, meaning the diagram

$$\begin{tikzcd}
A \arrow[rr, "\lambda_A"] \arrow[d, "(\mathrm{id}_A, D)"] & & \mathbb{W}(A) \arrow[d, "(\mathrm{id}_A, D)_*"] \\
A \ltimes M \arrow[r, "\lambda_A \oplus \lambda_M"] & \mathbb{W}(A) \ltimes \mathbb{W}(M) \arrow[r, "i_{1*}+i_{2*}"] & \mathbb{W}(A \ltimes M)
\end{tikzcd}$$

commutes. To see this, let $a \in A$: applying first (id_A, D), then $\lambda_A \oplus \lambda_M$

$$a \mapsto (a, Da) \mapsto (\lambda_A(a), \lambda_M(Da))$$

whose n^{th} Witt component is $(\lambda_{A,n}(a), \lambda_{M,n}(Da))$.

On the other hand, applying first λ_A and then $(\mathrm{id}_A, D)_*$ leads to an element with e^{th} Witt component $(\lambda_{A,e}(a), D\lambda_{M,e}(a))$. Because of Lem. 5.25 and the formula in Lem. 5.26 shows that the diagram commutes if and only if D is λ-linear. □

Recall that classically, Ḱahler differentials over a ring A are universal among the derivations over A, in the sense, that for a derivation $D : A \to M$ there is a unique map of A-modules $f : \Omega^1_A \to M$ such that $D = f \circ d$. Another way to express this is by saying the module of Ḱhler differentials Ω^1_A over A corepresents the functor that assigns to an A-module M the set of derivations $\mathrm{Der}(A, M)$. In the λ-world we have the following analogue.

Lemma 5.33. *Let (A, λ_A) be a λ-ring. There exists a derivation*

$$(A, \lambda_A) \xrightarrow{d} (\Omega^1_{(A,\lambda_A)}, \lambda_{\Omega^1_{(A,\lambda_A)}})$$

which corepresents the functor that to an (A, λ_A)-module (M, λ_M) assignes the set of derivations $\mathrm{Der}\left((A, \lambda_A), (M, \lambda_M)\right)$.

Proof. The target of the map: consider the free (A, λ_A)-module (F, λ_F) generated by the symbols $\{d(a) \mid a \in A\}$, and quotient out the relations that we would like to have: $d(a + b) - d(a) - d(b)$, $d(ab) - bd(a) - ad(b)$ and $\lambda_{F,n}(da) - \sum_{e|n} \lambda_{A,e}(a)^{\frac{n}{e}-1} d\lambda_{A,e}(a)$ for $a, b \in A$, $n \in \mathbb{N}$. The resulting object is denoted $(\Omega^1_{(A,\lambda_A)}, \lambda_{\Omega^1_{(A,\lambda_A)}})$.
The map: d takes a to the class of $d(a)$ under these relations.

By construction, for a λ-derivation $D : (A, \lambda_A) \to (M, \lambda_M)$ there is a unique well-defined map of λ-modules

$$f : (\Omega^1_{(A,\lambda_A)}, \lambda_{\Omega^1_{(A,\lambda_A)}} \to (M, \lambda_M)$$

such that $D = f \circ d$. □

The main theorem of this section identifies Ω^1_A and $\Omega^1_{(A,\lambda_A)}$ as A-modules via the canonical morphism given by the universal property of Ḱahler differentials.

Theorem 5.34. *For every λ-ring (A, λ_A) the canonical map*

$$\Omega^1_A \to \Omega^1_{(A,\lambda_A)}$$

is an A-module isomorphism.

Proof. Let

$$(\mathscr{A}/A)_{\mathrm{ab}} \underset{(-)_{\mathrm{ab}}}{\overset{i}{\rightleftarrows}} (\mathscr{A}/A) \qquad \text{and } (\mathscr{A}_\lambda/(A, \lambda_A))_{\mathrm{ab}} \underset{(-)_{\mathrm{ab}}}{\overset{i^\lambda}{\rightleftarrows}} (\mathscr{A}_\lambda/(A, \lambda_A))$$

be the forgetful functors (forgetting the abelian groups structurem together with their left adjoints. They fit into the following diagram

$$\begin{tikzcd}
\mathscr{A}/A \arrow[r, shift left, "(-)_{\mathrm{ab}}"] \arrow[d, shift left, "R_{(A,\lambda_A)}"] & (\mathscr{A}/A)_{\mathrm{ab}} \arrow[l, shift left, "i"] \arrow[r, shift left, "F"] \arrow[d, shift left, "R_{(A,\lambda_A)}"] & \mathscr{M}(A) \arrow[l, shift left, "G"] \arrow[d, shift left, "R'"] \\
\mathscr{A}_\lambda/(A, \lambda_A) \arrow[u, shift left, "U_{A,\lambda_A}"] \arrow[r, shift left, "(-)_{\mathrm{ab}}"] & (\mathscr{A}_\lambda/(A, \lambda_A))_{\mathrm{ab}} \arrow[u, shift left, "U_{(A,\lambda_A)}"] \arrow[l, shift left, "i^\lambda"] \arrow[r, shift left, "F^\lambda"] & \mathscr{M}(A, \lambda_A) \arrow[u, shift left, "U'"] \arrow[l, shift left, "G^\lambda"]
\end{tikzcd}$$

where in the right hand square the vertical funtors are adjoint equivalences, as we have seen. (This means that the composition of the top (resp. bottom) adjunctions of the whole square determine the top (resp. bottom) adjunctions of the left-hand square.)

Let $K = i \circ G$. Then we define a functor H, such that it gives rise to an adjunction $(H, K, \varepsilon, \eta)$. Recall what K does: it takes an A-module M to $f : A \ltimes M \to A$ (and the forgets $+_{A \ltimes M}, 0_{A \ltimes M}$ and $-_{A \ltimes M}$). Let H be the functor that assigns to a ring $f : B \to A$ over A the A-module $A \times_B \Omega^1_B$.

Similarly in the λ-world, we define a functor H^λ such that the composition $K^\lambda = i^\lambda \circ G^\lambda$ is its right adjoint: recall that K^λ takes an (A,λ_A)-module (M,λ_M) to the canonical projection $f : (A \ltimes M, \lambda_{A\ltimes M}) \to (A,\lambda_A)$ (and then forgets the abelian group object structure). Define H^λ to be the functor assigning to $f : (B,\lambda_B) \to (A,\lambda_A)$ the (A,λ_A)-module $(A,\lambda_A) \otimes_{(B,\lambda_B)} \Omega^1_{(b,\lambda_B)}$.

Thus we get a diagram of adjunctions

$$\begin{tikzcd}
\mathscr{A}/A \arrow[r, shift left, "H"] \arrow[d, shift left, "R_{(A,\lambda_A)}"] & \mathscr{M}(A) \arrow[l, shift left, "K"] \arrow[d, shift left, "R'"] \\
\mathscr{A}_\lambda/(A,\lambda_A) \arrow[r, shift left, "H^\lambda"] \arrow[u, shift left, "U_{A,\lambda_A}"] & \mathscr{M}(A,\lambda_A) \arrow[l, shift left, "K^\lambda"] \arrow[u, shift left, "U'"]
\end{tikzcd}$$

with the middle column "missing" from the above diagram. And this shows that up to unique natural isomorphism the composition of functors $R_{(A,\lambda_A)} \circ K$ coincides with the composition $K^\lambda \circ R$. And by uniqueness of the left adjoint, the same holds for the compositions $H \circ U_{(A,\lambda_A)}$ and $U' \circ H^\lambda$.

It follows that the canonical natural transformation

$$A \otimes_B \Omega^1_B \to U'\left((A,\lambda_A) \otimes_{(B,\lambda_B)} \Omega^1_{(B,\lambda_B)}\right)$$

is an isomorphism, and gives the desired result for $(B,\lambda_B) = (A,\lambda_A)$. □

This means, that for a λ-ring (A,λ_A) the A-module of usual differentials Ω^1_A the richer structure of an (A,λ_A)-module. In the case of the λ-ring $(\mathbb{W}(A),\Delta_A)$ this implies the existence of natural F_n-linear maps, that are also denoted $F_n : \Omega^1_{\mathbb{W}(A)} \to \Omega^1_{\mathbb{W}(A)}$.

Theorem 5.35. *Let A be a ring. There are natural F_n-linear maps $F_n : \Omega^1_{\mathbb{W}(A)} \to \Omega^1_{\mathbb{W}(A)}$ such that*

$$F_n(da) = \sum_{e|n} \Delta_{A,e}(a)^{\frac{n}{e}-1} d\Delta_{A,e}(a).$$

Moreover,

1. *for $m,n \in \mathbb{N}$: $F_mF_n = F_{nm}$ and $F_1 = \mathrm{id}$,*
2. *for $n \in \mathbb{N}$ and $a \in \mathbb{W}(A)$: $dF_n(a) = nF_n(da)$,*
3. *for $n \in \mathbb{N}$ and $a \in A$: $F_n(d[a]) = [a]^{n-1}d[a]$.*

Proof. We apply the previous theorem to the λ-ring $\mathbb{W}(A),\Delta_A)$ to get a canonical isomorphism

$$\Omega^1_{\mathbb{W}(A)} \xrightarrow{\sim} \Omega^1_{(\Omega_A,\Delta_A)}.$$

The crucial point is that the target of this map is a $(\mathbb{W}(A),\Delta_A)$-module, which comes together with a map $\lambda_{(\Omega_{\mathbb{W}(A)},\Delta_A)}$. We set

$$F_n = \lambda_{(\Omega_{\mathbb{W}(A)},\Delta_A),n} : \Omega^1_{(\mathbb{W}(A),\Delta_A)} \to \Omega^1_{(\mathbb{W}(A),\Delta_A)}$$

as the n^{th} Witt component of this map. It is obviously $F_n = w_n \circ \Delta_A$-linear and by the definition of a λ-derivation satisfies the given formula.

The identities follow with simple calculations. □

5.4 The big de Rham–Witt complex

The theme of the last section of this series is the existence of an initial object in the category of (big) Witt complexes — the big de Rham–Witt complex.

Theorem 5.36. *Let A be a (commutative unital) ring and S a truncation set. There is an initial Witt complex*

$$S \mapsto \mathbb{W}\,\Omega_S(A)$$

over the ring A. Moreover, for each degree q, the canonical map

$$\check{\Omega}^q_{\mathbb{W}_S(A)} \xrightarrow{\eta_S} \mathbb{W}_S\,\Omega^q_A$$

is surjective and we have commutative diagrams

$$
\begin{tikzcd}
\check{\Omega}^q_{\mathbb{W}_S(A)} \arrow[r, "\eta_S"] \arrow[d, "R^S_T"] & \mathbb{W}_S\,\Omega^q_A \arrow[d, "R^S_T"] \\
\check{\Omega}^q_{\mathbb{W}_T(A)} \arrow[r, "\eta_T"] & \mathbb{W}_T\,\Omega^q_A
\end{tikzcd}
\qquad
\begin{tikzcd}
\check{\Omega}^q_{\mathbb{W}_S(A)} \arrow[r, "\eta_S"] \arrow[d, "d"] & \mathbb{W}_S\,\Omega^q_A \arrow[d, "d"] \\
\check{\Omega}^{q+1}_{\mathbb{W}_S(A)} \arrow[r, "\eta_S"] & \mathbb{W}_S\,\Omega^{q+1}_A
\end{tikzcd}
\qquad
\begin{tikzcd}
\check{\Omega}^q_{\mathbb{W}_S(A)} \arrow[r, "\eta_S"] \arrow[d, "F_m"] & \mathbb{W}_S\,\Omega^q_A \arrow[d, "F_m"] \\
\check{\Omega}^q_{\mathbb{W}_{\frac{S}{m}}(A)} \arrow[r, "\eta_{\frac{S}{m}}"] & \mathbb{W}_{\frac{S}{m}}\,\Omega^q_A
\end{tikzcd}
$$

The maps on the left hand side in the diagrams from this statement have been defined in Lemma 5.15. It stands to reason to define the complex $\mathbb{W}_S\,\Omega_A$ as quotient of $\check{\Omega}_{\mathbb{W}_S(A)}$ in a way to make the diagrams commute. Furthermore, one defines Verschiebung as maps of graded abelian groups $\mathbb{W}_{\frac{S}{n}}\,\Omega_A \xrightarrow{V_n} \mathbb{W}_S\,\Omega_A$ such that

$$
\begin{tikzcd}
\mathbb{W}_{\frac{S}{n}}(A) \arrow[r, "\eta_{\frac{S}{n}}"] \arrow[d, "V_n"] & \mathbb{W}_{\frac{S}{n}}\,\Omega^0_A \arrow[d, "V_n"] \\
\mathbb{W}_S(A) \arrow[r, "\eta_S"] & \mathbb{W}_S\,\Omega^0_A
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{W}_{\frac{S}{n}}\,\Omega_A \arrow[r, "V_n"] \arrow[d, "R^{\frac{S}{n}}_{\frac{T}{n}}"] & \mathbb{W}_S\,\Omega_A \arrow[d, "R^S_T"] \\
\mathbb{W}_{\frac{T}{n}}\,\Omega_A \arrow[r, "V_n"] & \mathbb{W}_T\,\Omega_A
\end{tikzcd}
$$

$$
\begin{tikzcd}
& \mathbb{W}_{\frac{S}{n}}\,\Omega_A \otimes \mathbb{W}_S\,\Omega_A \arrow[dl, "\mathrm{id}\otimes F_n"'] \arrow[dr, "V_n\otimes \mathrm{id}"] & \\
\mathbb{W}_{\frac{S}{n}}\,\Omega_A \otimes \mathbb{W}_{\frac{S}{n}}\,\Omega_A \arrow[d, "\mu"] & & \mathbb{W}_S\,\Omega_A \otimes \mathbb{W}_S\,\Omega_A \arrow[d, "\mu"] \\
\mathbb{W}_{\frac{S}{n}}\,\Omega_A \arrow[rr, "V_n"] & & \mathbb{W}_S\,\Omega_A
\end{tikzcd}
$$

commute.

The definition of $\mathbb{W}_S\,\Omega_A$ and V_n will be done, as S ranges over all finite truncation sets (which we have seen to suffice), $T \subset S$ over all subtruncation sets, and n over all natural numbers, by induction on the cardinality of S. Then one can show that the object obtained together with this structure actually is a big Witt complex and moreover that it is the initial one.

Proof. To start the induction, let $S = \emptyset$, and define $\mathbb{W}_\emptyset\,\Omega_A$ to be the terminal graded ring which is zero in al degrees, and let

$$\eta_\emptyset : \check{\Omega}_{\mathbb{W}_\emptyset(A)} \to \mathbb{W}_\emptyset\,\Omega_A$$

to be the unique map of graded rings. The maps $R^\emptyset_\emptyset$, F_n, d, and V_n are trivial as well.
Now let S be a finite truncation set, and assume that for all proper truncation sets $T \subsetneq S$, and $U \subset T$ and all $n \in \mathbb{N}$ the maps η_T, R^T_U,F_n, d, and V_n have been defined such that the desired properties are satisfied.

Let N_S be the graded ideal of $\check{\Omega}_{\mathbb{W}_S(A)}$ generated by all sums of the form

$$\sum_\alpha V_n(x_\alpha)dy_{1,\alpha}\cdots dy_{q,\alpha} \quad \text{and} \quad d\left(\sum_\alpha V_n(x_\alpha)dy_{1,\alpha}\cdots dy_{q,\alpha}\right),$$

where $x_\alpha \in \mathbb{W}_{\frac{S}{n}}(A)$ and $y_{1,\alpha}, \dots y_{q,\alpha} \in \mathbb{W}_S(A)$ and $n \geqslant 2$, $q \geqslant 1$ such that the projection of the sum

$$\eta_{\frac{S}{n}}\left(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\cdots dy_{q,\alpha}\right)$$

to $\mathbb{W}_{\frac{S}{n}}\,\Omega_A^q$ is zero. Let

$$\mathbb{W}_S\,\Omega_A = \check{\Omega}_{\mathbb{W}_S(A)}/N_S$$

be the quotient, and η_S the quotient map.
Next we define $V_n : \mathbb{W}_{\frac{S}{n}}\,\Omega_A \to \mathbb{W}_S\,\Omega_A$, which has to "commute" with η_S and $\eta_{\frac{S}{n}}$ as map of graded abelian groups by

$$V_n\eta_{\frac{S}{n}}(xF_n dy_1\cdots F_n dy_q) = \eta_S(V_n(x)dy_1\cdots dy_q)$$

which defines V_n uniquely in that every element of $\mathbb{W}_{\frac{S}{n}}\,\Omega_A^q$ can be written as a sum of elements $\eta_{\frac{S}{n}}(XF_n dy_1\cdots dy_q)$ with $x \in \mathbb{W}_{\frac{S}{n}}(A)$ and $y_i \in \mathbb{W}_S(A)$.
We come to the existence and uniqueness of the maps R_T^S, d and F_n, which make the diagrams in the theorem commute. Note that once existence is established, uniqueness is clear due to the commutativity of these diagrams. For the existence, we have to show that applying the left hand vertical maps R_T^S, d and F_n to the q-graded piece of the kernel N_S^q of $\check{\Omega}^q_{\mathbb{W}_S(A)}$ is trivial in the quotient. More precisely, we have to show

$$\begin{aligned}
\eta_T(R_T^S(N_S^q)) &= 0\\
\eta_S(d(N_S^q)) &= 0\\
\eta_{\frac{S}{m}}(F_m(N_S^q)) &= 0
\end{aligned}$$

One has to use the properties established for the maps on $\check{\Omega}$. Let for $n \in \mathbb{N}$

$$\omega = \sum_\alpha V_n(X_\alpha)dy_{1,\alpha}\cdots dy_{q,\alpha} \in \check{\Omega}^q_{\mathbb{W}_S(A)}$$

such that $0 = \eta_{\frac{S}{n}}\left(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\dots F_n dy_{q,\alpha}\right) \in \mathbb{W}_{\frac{S}{n}}\,\Omega_A^q$ (this defines a general element of the kernel) and show that

$$\begin{aligned}
\eta_T R_S^T(\omega) &= 0\\
\eta_S(dd\omega) &= 0\\
\eta_{\frac{S}{m}}F_m(\omega) &= 0\\
\eta_{\frac{S}{m}}F_m(d\omega) &= 0
\end{aligned}$$

Rewriting $R_S^T(\omega)$, to show that

$$\eta_T R_S^T(\omega) = \eta_T\left(\sum_\alpha V_n R^{\frac{S}{n}}_{\frac{S}{n}} dR_T^S(y_{1,\alpha})\cdots dR_T^S(y_{q,\alpha})\right)$$

it is enough to show that the following element is zero:

$$\begin{aligned}
\eta_{\frac{T}{n}}\left(\sum_\alpha R^{\frac{S}{n}}_{\frac{T}{n}}(x_\alpha)F_n dR_T^S(y_{1,\alpha})\cdots F_n dR_T^S(y_{q,\alpha})\right) &= \eta_{\frac{T}{n}} R^{\frac{S}{n}}_{\frac{T}{n}}\left(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\cdots F_n dy_{q,\alpha}\right)\\
&= R^{\frac{S}{n}}_{\frac{T}{n}}\eta_{\frac{S}{n}}\left(\sum_\alpha x_\alpha F_n dy_{1,\alpha}\cdots dy_{q,\alpha}\right) \quad \text{by induction hypothesis}\\
&= 0 \quad \text{by induction hypothesis}
\end{aligned}$$

The proofs of the remaining equalities will be left as an exercise.
To complete the definition/construction of $\mathbb{W}_S\,\Omega_A$ together with the maps η_S, R_T^S, d, F_n and V_n, it remains to verify that the three diagrams (two squares and one pentagon) commute.

The diagram

$$\begin{tikzcd}
\mathbb{W}_{\frac{S}{n}}(A) \arrow[r, "\eta_{\frac{S}{n}}"] \arrow[d, "V_n"] & \mathbb{W}_{\frac{S}{n}}\,\Omega_A^0 \arrow[d, "V_n"]\\
\mathbb{W}_S(A) \arrow[r, "\eta_S"] & \mathbb{W}_S\,\Omega_A^0
\end{tikzcd}$$

commutes by definition of the Verschiebung.

The diagram

$$\begin{tikzcd} \mathbb{W}_{\frac{S}{n}}\,\Omega_A \arrow[r, "V_n"] \arrow[d, "R^{\frac{S}{n}}_{\frac{T}{n}}"] & \mathbb{W}_S\,\Omega_A \arrow[d, "R^S_T"] \\ \mathbb{W}_{\frac{T}{n}}\,\Omega_A \arrow[r, "V_n"] & \mathbb{W}_T\,\Omega_A \end{tikzcd}$$

commutes by the following calculation, taking into account that every element of $\mathbb{W}_{\frac{S}{n}}$ can be written as a sum of elements of the form $\eta_{\frac{S}{n}}(xF_n dy_1 \cdots dy_q)$ with $x \in \mathbb{W}_{\frac{S}{n}}(A)$ and $y_i \in \mathbb{W}_S(A)$:

$$\begin{aligned} R^S_T V_n \eta_{\frac{S}{n}}(xF_n dy_1 \cdots dy_q) &= R^S_T \eta_S(V_n(x) dy_1 \cdots dy_q) && \text{by definition of } V_n \\ &= \eta_T R^S_T(V_n(x) dy_1 \cdots dy_q) && \text{by definition of } R^S_T \\ &= \eta_T(V_n R^{\frac{S}{n}}_{\frac{T}{n}}(x) dR^S_T(y_1) \cdots dR^S_T(y_q)) && \text{by induction hypothesis} \\ &= V_n \eta_{\frac{T}{n}}(R^{\frac{S}{n}}_{\frac{T}{n}}(x) F_n dR^S_T(y_1) \cdots F_n dR^S_T(y_q)) && \text{by definition of } V_n \\ &= V_n R^{\frac{S}{n}}_{\frac{T}{n}} \eta_{\frac{S}{n}}(xF_n dy_1 \cdots dy_q) && \text{by definition of } R^{\frac{S}{n}}_{\frac{T}{n}} \end{aligned}$$

The commutativity of the pentagon is discussed in the exercises.
The next point is to check that what we just defined is indeed a Witt complex over A.As a reminder, for this is needed: $V_1 = \mathrm{id}$, $V_n V_m = V_{nm}$, $F_n V_m = n\,\mathrm{id}$ and $F_m V_n = V_n F_m$ if $(nm) = 1$. The first is clear by definition. For the second identity compute

$$\begin{aligned} V_{mn}\eta_{\frac{S}{mn}}(xF_{mn}dy_1 \cdots F_{mn}dy_q) &= \eta_S(V_{mn}(x)dy_1 \cdots dy_q) && \text{by definition of } V_{mn} \\ &= \eta_S(V_m(V_n(x))dy_1 \cdots dy_q) && \text{by the desired equation on } \mathbb{W}(A) \\ &= V_m \eta_{\frac{S}{m}}(V_n(x)F_m dy_1 \cdots F_m dy_q) && \text{by definition of } V_m \\ &= V_m(V_n(\eta_{\frac{S}{mn}}(x))F_m d\eta_S(y_1) \cdots F_m d\eta_S(y_q)) && \text{by existence of } F_m \text{ with } \eta_{\frac{S}{m}} F_m = F_m \eta_S \\ &= V_m(V_n(\eta_{\frac{S}{mn}}(x)F_{mn} d\eta_S(y_1) \cdots F_{mn} d\eta_S(y_q))) && \text{by inductive hypothesis} \\ &= V_m(V_n \eta_{\frac{S}{mn}}(xF_{mn}dy_1 \cdots F_{mn}dy_q)) && \text{by definition of } F_{mn} \end{aligned}$$

Similarly for the third identity:

$$\begin{aligned} F_n V_n \eta_{\frac{S}{n}}(xF_n dy_1 \cdots dy_q) &= F_n \eta_S(V_n(x)dy_1 \cdots dy_q) && \text{by definition of } V_n \\ &= \eta_{\frac{S}{n}} F_n(V_n(x)dy_1 \cdots dy_q) && \text{by definition of } F_n \\ &= n\eta_{\frac{S}{n}}(xF_n dy_1 \cdots dy_q) && \text{by induction} \end{aligned}$$

The fourth identity will be discussed in the exercises.
Finally, we have to show that the complex which we constructed is initial among Witt complexes over A. To this end, let $E^\bullet_S$ be a Witt complex over A together with the map

$$\eta^E_S : \check{\Omega}_{\mathbb{W}_S(A)} \to E^\bullet_S$$

which was constructed earlier. One has to show that this map factors through $\mathbb{W}_S\,\Omega_A$

$$\begin{tikzcd} \check{\Omega}_{\mathbb{W}_S(A)} \arrow[rr, "\eta^E_S"] \arrow[dr, "\eta_S"'] & & E^\bullet_S \\ & \mathbb{W}_S\,\Omega_A \arrow[ur, dotted] & \end{tikzcd}$$

Since η_S is by construction surjective, the map f_S has to be unique if it exists. To show existence, by the same reasoning as before, we may assume that the truncation set S is finite, and proceed again by induction on the cardinality of S, the case $S = \emptyset$ being easy, as it is simply the identity. Thus let S be a finite truncation set, and assume that for every proper subtruncation set $T \subsetneq S$, the factorisation $\eta_T^E = f_T \eta_T$ exists. The proceeding is now similar to the existence of the maps R_T^S, F_n, d, as we have to show again, that for any $n \in \mathbb{N}$, $x_\alpha \in \mathbb{W}_{\frac{S}{n}}(A)$ and $y_{1,\alpha}, \ldots, y_{y,\alpha} \in \mathbb{W}_S(A)$ such that

$$\eta_{\frac{S}{n}} \left(\sum_\alpha x_\alpha F_n dy_{1,\alpha} \cdots F_n dy_{q,\alpha} \right) \in \mathbb{W}_{\frac{S}{n}} \Omega_A^q$$

vanishes, the element

$$\eta_S^E \left(\sum_\alpha V_n(x_\alpha) dy_{1,\alpha} \cdots dy_{q,\alpha} \right) \in E_S^q$$

vanishes as well.

Using that $E_S^\bullet$ is a Witt complex, we find (with some intermediate steps that are omitted) with the inductive hypothesis that

$$\eta_S^E \left(\sum_\alpha V_n(x_\alpha) dy_{1,\alpha} \cdots dy_{q,\alpha} \right) = V_n f_{\frac{S}{n}} \eta_{\frac{S}{n}} \left(\sum_\alpha x_\alpha F_n dy_{1,\alpha} \cdots F_n dy_{q,\alpha} \right)$$

which vanishes by induction.

This is the induction step to get the factorisation for S.

FInally, one has to show that the so obtained maps f_S for varying S constitute a map of Witt complexes, which means that it commutes with the respective d's, F_n's and V_n's. We have seen in Corollary 5.16 that the maps η^E commute with Frobenius, more precisely for $m \in \mathbb{N}$

$$F_m \circ \eta_S^E = \eta_{\frac{S}{m}} \circ F_m$$

and by construction, the same holds true for the maps η in $\mathbb{W}\,\Omega$. It follows that

$$F_m \circ f_S = f_{\frac{S}{m}} \circ F_m$$

for all $m \in \mathbb{N}$. Likewise, since η and η^E commute with the differentials d, the maps f_S are bound to do so as well. Finally, it remains to show that for every truncation set S and for every positive integer m, one has $f_S \circ V_m = V_m \circ f_{\frac{S}{m}}$: again by the reasoning that every element of $\mathbb{W}_{\frac{S}{m}}$ can be written as a sum of elements of the form $\eta_{\frac{S}{m}}(xF_n dy_1 \cdots dy_q)$ with $x \in \mathbb{W}_{\frac{S}{m}}(A)$ and $y_i \in \mathbb{W}_S(A)$:

$$\begin{aligned}
f_S V_m \eta_{\frac{S}{m}}(x \cdot F_m dy_1 \cdots F_m dy_q) &= f_S \eta_S(V_m(x) \cdot dy_1 \cdots dy_q) \quad \text{by definition of } V_n \\
&= \eta_S^E(V_m(x) \cdot dy_1 \cdots dy_q) \quad \text{by factorisation of } \eta^E \\
&= \eta_S^E(V_m(x)) \cdot \eta_S^E(dy_1 \cdot dy_q) \quad \text{by multiplicativity of } \eta^E \\
&= V_m(\eta_{\frac{S}{m}}^E(x)) \cdot \eta_S^E(dy_1 \cdots dy_q) \quad \text{since } V_m \text{ and } \eta^E \text{ commute in degree zero} \\
&= V_m(\eta_{\frac{S}{m}}^E(x) \cdot F_m \eta_S^E(dy_1 \cdots dy_m)) \quad \text{by definition} \\
&= V_m(\eta_{\frac{S}{m}}^E(x) \cdot \eta_{\frac{S}{m}}^E F_m(dy_1 \cdots dy_m)) \quad \text{since } \eta^E \text{ and } F_m \text{ commute} \\
&= V_m(\eta_{\frac{S}{m}}^E(x \cdot F_m dy_1 \cdots dy_m) \quad \text{by multiplicativity of } \eta^E \\
&= V_m f_{\frac{S}{m}} \eta_{\frac{S}{m}}(x \cdot F_m dy_1 \cdots F_m dy_q) \quad \text{by factorisation of } \eta^E
\end{aligned}$$

This completes the proof of the theorem. □

Definition 5.37. The initial Witt complex $\mathbb{W}_S\,\Omega_A$ is called the big de Rham–Witt complex for the truncation set S of A. If $S = \mathbb{N}$, it is denotes by $\mathbb{W}\,\Omega_A$ and called the big de Rham–Witt complex of A.

It is clear by definition, that considering the unit truncation set, one obtains the usual de Rham complex. More precisely,the map

$$\eta_{\{1\}} : \Omega_A^q \xrightarrow{\sim} \mathbb{W}_{\{1\}}\,\Omega_A^q$$

is an isomorphism for all q. Moreover, in degree zero, one has an isomorphism

$$\eta_S : \mathbb{W}_S(A) \to \mathbb{W}_S\, \Omega^0_A$$

for all truncation sets S. This is in line with the p-typical de Rham–Witt complex.

It is possible to define a relative version of the big de Rham–Witt complex, using relative λ-derivations. This is a big version of Langer and Zink's relative de Rham–Witt complex [10].

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