11.1 Let $f : (A, \lambda_A) \to (B, \lambda_B)$ e a morphism of λ-rings. Describe the induced funtors f_* and f^* on λ-modules.

The category of (A, λ_A)-modules is functorial in the base ring in the sense that a map of λ-rings

$$f : (A, \lambda_A) \to (B, \lambda_b)$$

induces a functor

$$f_* : \mathscr{M}(A, \lambda_A) \to \mathscr{M}(B, \lambda_B)$$

where an (A, λ_A)-module (N, λ_N) is seen as a (B, λ_B)-module $f_*(N, \lambda_N)$ via f. Its left adjoint sends a (B, λ_B)-module (M, λ_M) to the (A, λ_A)-module

$$f^*(M, \lambda_M) = (A, \lambda_A) \otimes_{(B,\lambda_B)} (M, \lambda_M)$$

where the tensor product is defined as $A \otimes_B M$ on the module and the λ-operation $\lambda_{A\otimes_B M}$ associated to it is the composition

$$A \otimes_B M \xrightarrow{\lambda_A \otimes_{\lambda_B} \lambda_M} \mathbb{W}(A) \otimes \mathbb{W}(B)\, \mathbb{W}(M) \xrightarrow{a\otimes x \mapsto (w_n(a)\otimes x_n)_n} \mathbb{W}(A \otimes_B M)$$

11.2 Let (A, λ_A) be a λ-ring. Show that the category $\mathscr{M}(A, \lambda_A)$ of (A, λ_A)-modules is abelian.

Let (A, λ_A) be a λ-ring, M an A-module and $\lambda_M : M \to \mathbb{W}(M)$ a map. Then (M, λ_M) is a (A, λ_A)-module if and only if the components $\lambda_{M,n} = w_n \circ \lambda_M : M \to M$ are $\psi_{A,n} = w_n \circ \lambda_A$-linear and satisfy

$$\lambda_{M,1} = \mathrm{id} \quad \text{and} \quad \lambda_{M,n} \circ \lambda_{M,m} = \lambda_{M,nm}$$

It follows that we may identify the category $\mathscr{M}(A, \lambda_A)$ of (A, λ_A)-modules with the category $\mathscr{M}(A^\psi[\mathbb{N}])$ of left modules over the twisted monoid algebra $A^\psi[\mathbb{N}]$

$$\mathscr{M}(A, \lambda_A) \leftrightarrow \mathscr{M}(A^\psi[\mathbb{N}])$$

associating to the (A, λ_A)-module (M, λ_M) the A-module M with n acting through $\lambda_{M,n} : M \to M$. In particular, the catgory $\mathscr{M}(A, \lambda_A)$ is abelian.

11.3 Find a counter example of a λ-ring (A, λ_A) to show that in general (A, λ_A) is not an (A, λ_A)-module.

11.4 Show that the functors H and K from the proof of the main theorem on λ-derivations form an adjuntion.

Recall that K takes an A-module M to $f : A \ltimes M \to A$ (and then forgets $+_{A\ltimes M}, 0_{A\ltimes M}$ and $-_{A\ltimes M}$), and H assigns to a ring $f : B \to A$ over A the A-module $A \times_B \Omega_B$.

Let ε and η be the natural transformations given by

$$\begin{aligned} \varepsilon(1 \otimes d(a, x)) &= x \\ \eta(b) &= (f(b), 1 \otimes db) \end{aligned}$$

Then

$$H \xRightarrow{H\circ\eta} H \circ K \circ H \xRightarrow{\varepsilon\circ H} H$$

$$K \xRightarrow{\eta\circ K} K \circ H \circ K \xRightarrow{K\circ\varepsilon} K$$

are equal to the identity transformations : $H \circ \eta$ maps $a \otimes db$ from $H(f : B \to A)$ to $a \otimes d(f(b), 1 \otimes db)$ in $(H \circ K \circ H)(f : B \to A)$ and $\epsilon \circ H$ maps this to $a \cdot (1 \otimes b) = a \otimes b$ in $H(f : B \to A)$.

Similarly for the second diagram.