9.1 Recall the definition of a (special) λ-ring due to Grothendieck, and show that it coincides with the definition given in the lecture.

We take the definition from [1, I. §], who give Grothendieck's original defintions as follows.

Definition 9.1. A special λ-ring is a unital commutative ring A nd a countable set of maps $\lambda^n : A \to A$ such that the following is satisfied :

1. $\lambda^0(x) = 1$
2. $\lambda^1(x) = x$
3. $\lambda^n(x+y) = \sum_{r=0}^{n} \lambda^r(x)\lambda^{n-r}(y)$

If t is an indeterminte, define for $x \in R$

4. $\lambda_t(x) = \sum_{n\in\mathbb{N}_0} \lambda^n(x)t^n$

Then the relations (1) and (3) show that λ_t is a homomorphism form the additive group A to the multplicative group $1 + A[\![t]\!]$ i.e.

5. $\lambda_t(x+y) = \lambda_t(x)\lambda_t(y)$

According to (2), λ_t is a right inverse of the homomorphism

$$1 + \sum_{n\in\mathbb{N}} x_n t^n \mapsto x_1$$

in particular λ_t is injective.

We see that it is sufficient to give the data $(A, \lambda_t : A \to 1 + A[\![t]\!])$. The group $1 + A[\![t]\!]$ is denoted for brevity by $\Lambda(A)$. Now the additive group of the ring of big Witt vectors $\mathbb{W}(A)$ is naturally isomorphic to $\Lambda(A)$. In fact the following fact holds.

Proposition 9.2. *The diagram of natural group homomorphisms*

$$\begin{array}{ccc} \mathbb{W}(A) & \xrightarrow{\gamma} & \Lambda(A) \\ {\scriptstyle w}\downarrow & & \downarrow{\scriptstyle t\frac{d}{dt}\log} \\ A^{\mathbb{N}} & \xrightarrow{\gamma^w} & tA[\![t]\!], \end{array}$$

with $\gamma(a_1, a_2, \ldots) = \prod_{n\in\mathbb{N}}(1 - a_n t^n)^{-1}$ *and* $\gamma^w(x_1, x_2, \ldots) = \sum_{n\in\mathbb{N}} x_n t^n$*, commutes and the horizontal maps are isomorphisms.*

The set $\Lambda(A)$ can be considered as a ring with different ring structures. We want it to be characterised by being natural in A, addition should be given by power series multiplication (in line with what we said earlier). The product should satisfy a formula where we have four different choices of signs

$$(1 \pm at)^{\pm 1} * (1 \pm bt)^{\pm 1} = (1 \pm abt)^{\pm 1}$$

The four different rings $\Lambda(A)_{\pm\pm}$ are naturally isomorphic and the choice $--$ makes

$$\gamma : \mathbb{W}(A) \to \Lambda(A)$$

into a ring isomorphism. We denote by $u_{\pm\pm}$ the natural ring isomorphisms

$$u_{\pm\pm} : \Lambda(A) = \Lambda(A)_{--} \to \Lambda(A)_{\pm\pm}$$

and $\gamma_{\pm\pm} = u_{\pm\pm} \circ \gamma$.

In the original definition of special λ-rings the choice $++$ is used. There is a natural ring homomorphism

$$\begin{array}{rcl} \epsilon_{t,A} : \Lambda(A)_{++} & \to & A \\ 1 + a_1 t + \cdots & \mapsto & a_1 \end{array}$$

With this $\Lambda(A)_{++}$ is a λ-ring with the λ-operation

$$\Delta_{t,A} : \Lambda(A)_{++} \to \Lambda(\Lambda(A)_{++})_{++}$$

given by the unique natural ring homomorphism that is a section of $\epsilon_{t,\Lambda(A)_{++}}$ and satisfies for all $a \in A$

$$\Delta_{t,A}(1 + at) = 1 + (1 + at_2)t_1$$

Similar to the definition in the lecture with $\mathbb{W}(A)$ in place of $\Lambda(A)_{++}$, it is a functor from commutative rings to itself such that the following diagrams commute

$$\begin{array}{ccccc} \Lambda(A)_{++} & \xleftarrow{\epsilon_{t,\Lambda(A)_{++}}} & \Lambda(\Lambda(A)_{++})_{++} & \xrightarrow{\Lambda(\epsilon_{t,A})_{++}} & \Lambda(A)_{++} \\ & \diagdown\!\!\diagdown & \uparrow{\scriptstyle \Delta_{t,A}} & /\!\!/ & \\ & & \Lambda(A)_{++} & & \end{array}$$

and

$$\begin{array}{ccc} \Lambda(\Lambda(\Lambda(A)_{++})_{++})_{++} & \xleftarrow{\Delta_{t,\Lambda(A)_{++}}} & \Lambda(\Lambda(A)_{++})_{++} \\ \uparrow{\scriptstyle \Lambda(\Delta_{t,A})_{++}} & & \uparrow{\scriptstyle \Delta_{t,A}} \\ \Lambda(\Lambda(A)_{++})_{++} & \xleftarrow{\Delta_{t,A}} & \Lambda(A)_{++}. \end{array}$$

These diagrams express that the triple $(\Lambda(-)_{++}, \Delta_t, \epsilon_t)$ is a commonad in the category of commutative rings.

With this in mind we can give an equivalent definition of special λ-rings.

Definition 9.3. A special λ-ring is a pair (A, λ_t) of a ring A and a ring homomorphism $\lambda_t : A \to \Lambda(A)_{++}$ such that

$$\begin{array}{ccc} A & \xleftarrow{\epsilon_{t,A}} & \Lambda(A)_{++} \\ & \diagdown\!\!\diagdown & \uparrow{\scriptstyle \lambda_t} \\ & & A \end{array}$$

and

$$\begin{array}{ccc} \Lambda(\Lambda(A)_{++})_{++} & \xleftarrow{\Delta_{t,A}} & \Lambda(A)_{++} \\ \uparrow{\scriptstyle \Lambda(\lambda_t)_{++}} & & \uparrow{\scriptstyle \lambda_t} \\ \Lambda(A)_{++} & \xleftarrow{\lambda_t} & A \end{array}$$

commute. Morphisms are defined in the obvious way, similar to the defintion in class.

The commutativity of the diagrams express that (A, λ_t) is a coalgebra over the commonad $(\Lambda(-)_{++}, \Delta_t\epsilon_t)$.

This means, that in order to show that the definition of λ-rings in the lecture and the definition of special λ-rings here coincide, one has to show that the natural ring isomorphism $\gamma_{++} = u_{++} \circ \gamma$ induces an isomorphism of comonads

$$\gamma_{++} : (\mathbb{W}(-), \Delta, \epsilon) \to (\Lambda(-)_{++}, \Delta_t\epsilon_t)$$

in the sense that if (A, λ) is a coalgebra over $(\mathbb{W}(-), \Delta, \epsilon)$ then $(A, \gamma_{++}\circ\lambda)$ is a coalgebra over $(\Lambda(-)_{++}, \Delta_t\epsilon_t)$.

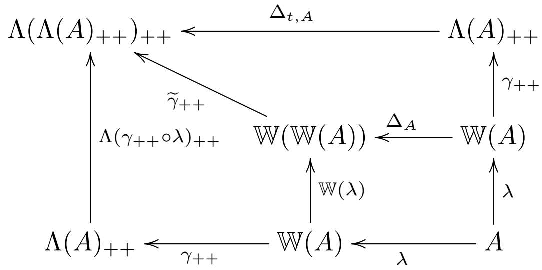
Indeed, because $\epsilon_A = w_1 : \mathbb{W}(A) \to A$ and the definition of $\epsilon_{t,A}$ from above, it is clear that we have a commutative diagram

$$\begin{array}{ccc} A & \xleftarrow{\epsilon_{t,A}} & \Lambda(A)_{++} \\ \Vert & & \uparrow{\scriptstyle \gamma_{++}} \\ A & \xleftarrow{\epsilon_A} & \mathbb{W}(A) \\ & \diagdown\!\!\diagdown & \uparrow{\scriptstyle \lambda} \\ & & A. \end{array}$$

Moreover we have a commutative diagram

$$\begin{array}{ccc} \Lambda(\Lambda(A)_{++})_{++} & \xleftarrow{\gamma_{++}} & \mathbb{W}(\Lambda(A)_{++}) \\ \uparrow{\scriptstyle \Lambda(\gamma_{++})_{++}} & & \uparrow{\scriptstyle \mathbb{W}(\gamma_{++})} \\ \Lambda(\mathbb{W}(A))_{++} & \xleftarrow{\gamma_{++}} & \mathbb{W}(\mathbb{W}(A)) \end{array}$$

thus a natural morphism $\widetilde{\gamma}_{++} : \mathbb{W}(\mathbb{W}(A)) \to \Lambda(\Lambda(A)_{++})_{++}$. Now consider the following diagram



where the small inner square commutes by hypothesis, the upper square commutes because of the characterisation of Δ_t via the formula

$$\Delta_{t,A}(1 + at) = 1 + (1 + at_2)t_1$$

and the euqivalent formula fomula for Δ_A :

$$\Delta_A([a]) = [[a]]$$

and the left square commutes by functoriality.

This shows the claim. For each n, the λ^n from the above definition is called n^{th} exterior operation associated to (A, λ). However, it should be noted, which is cler from the defitions, that this is not the same as the n^{th} Witt component λ_n of λ.

Références

[1] M.F. Atiyah and D.O. Tall. Group representations, Î»-rings and the j-homomorphism. *Topology*, 8(3) :253 – 297, 1969.