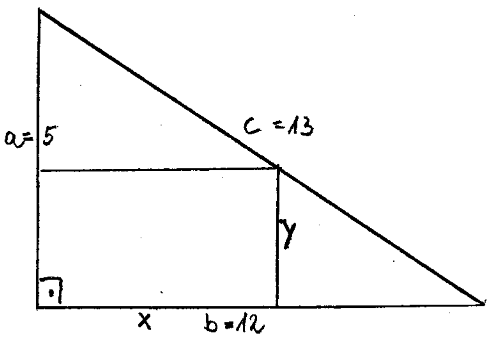
Quiz #6

Name: [redacted] Solution
StudentID:
Major:
Time: 20 minutes.

Justify your solutions and show all your steps. Write down formulae used. Make sure to look on the back side of this sheet.
Consider a right triangle as sketched with sides $a = 5$, $b = 12$, $c = 13$. You want to inscribe a rectangle as indicated with sides x and y. We will find out how to choose x and y, to maximize the area of the rectangle.

1. Write down the equation of the area $A(x, y)$ of the rectangle depending on x and y.



$A(x,y) = x \cdot y$ (= formula for area of rectangle)

2. Express y in terms of x.

Since for equivalent triangles the ratios of sides coincides, we get:

$$\frac{a}{b} = \frac{y}{b-x} \iff y = \frac{a}{b}(b-x) = a - \frac{a}{b}x$$

$$\frac{5}{12} = \frac{y}{12-x} \iff y = \frac{5}{12}(12-x) = 5 - \frac{5}{12}x$$

3. Express the area of the rectangle in terms of x (that is only depending on the variable x). You will get a function $A(x)$. (Answer: $A(x) = 5x - \frac{5}{12}x^2$)

$$A(x) = A\left(x,\ y = 5 - \frac{5}{12}x\right) = x \cdot \left(5 - \frac{5}{12}x\right) = 5x - \frac{5}{12}x^2$$

4. Find the first and second derivative of the function $A(x)$.

$$A'(x) = 5 - \frac{5}{6}x$$

$$A''(x) = -\frac{5}{6} < 0$$

5. Use this to find the maximal and/or minimal point(s) of $A(x)$.

$$0 = A'(x) = 5 - \frac{5}{6}x \quad \Leftrightarrow \quad 5 = \frac{5}{6}x \quad \Leftrightarrow \quad x = 6$$

This is a maximum because $A''(6) = -\frac{5}{6} < 0$

$$A(6) = 5 \cdot 6 - \frac{5}{12} \cdot 36 = 15 \qquad \Rightarrow \text{Maximal point of } A(x): \underline{\underline{P = (6, 15)}}$$

6. How do you have to choose x and y to maximize the area of the rectangle?

$A(x)$ has a maximum for $\underline{\underline{x = 6.}}$

In this case $y = 5 - \frac{5}{12} \cdot 6 = \underline{\underline{2.5}}$

7. What is this maximal area?

$A(x)$ has a maximum for $x = 6$.

In this case $\underline{\underline{A(6) = 15}}$

Note: To practice, solve the same questions for:

$a = 3,\ b = 4,\ c = 5$

$a = 15,\ b = 8,\ c = 17$

$a = 35,\ b = 12,\ c = 37$

$a = 21,\ b = 20,\ c = 29$

$a = 7,\ b = 24,\ c = 25$