Differentiation

The first derivative at a point calculates the slope of a curve at that point. The slope oif th curve at a point is the slope of the tangent line. We can find it using limits.

Definition 1. Let f be a function and x_0 an element of the domain. If the limit

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists, we call it first derivative of f at the point x_0 and denote it by $f'(x_0)$

Sometimes the derivative is denoted as

$$\frac{df}{dx}(x) \quad \text{or} \quad D_x f(x)$$

We say that f is differentiable at x_0, if the above limit exists. If it is differentiable at all point of the domain, we say f is differentiable. In that case, we are looking for a function $f' : x \mapsto m(x)$ where $m(x)$ is the slope of f at x.

Theorem 2. *If a function is differentiable at a point, it is also continuous at this point.*

But the converse is not true! If a function is discontinuous, it is not differentiable at this point. It is alos not differentible at a cusp or a pole.

Differentiation rules

1. Constant function rule: If $f(x) = k$ for $k \in \mathbb{R}$ then
$$f'(x) = 0$$
2. Identity function rule: If $f(x) = x$ then
$$f'(x) = 1$$
3. Power rule: For $f(x) = x^n$ with $n \in \mathbb{N}$
$$f'(x) = nx^{n-1}$$
4. Constant multiple rule: If $k \in \mathbb{R}$ and $f(x)$ is $f'(x)$ exists then
$$(kf(x))' = kf'(x)$$
5. Sum and difference rule: If f and g are differentiable then
$$(f+g)'(x) = f'(x) + g'(x)$$
6. Square root rule: If $f(x) = \sqrt{x}$ then
$$f'(x) = \frac{1}{2\sqrt{x}}$$
7. Product rule: If f and g are differentiable then
$$(f \cdot g)'(x) = f'(x)g(x) + g'(x)f(x)$$
8. Quotient rule: If f and g are differentiable and $g(x) \neq 0$ then
$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}$$

9. Trigonometric functions:

$$\begin{aligned}
\sin'(x) &= \cos(x) \\
\cos'(x) &= -\sin(x) \\
\tan'(x) &= \sec^2(x) \\
\sec'(x) &= \sec(x)\tan(x) \\
\cot'(x) &= -\csc^2(x) \\
\csc'(x) &= -\csc(x)\cot(x)
\end{aligned}$$

10. Chain rule: Let f and g be differentiable. Then

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x)$$

Higher derivatives

To get the second derivative, you need the first derivative first. THe process is recursive.

Implicit differentiation

Given an equation $f(y) = g(x)$. This can be seen as a function $y(x)$ given implicitely. Oftentimes we cannot solve directly for y in terms of x. But we can try to differentiate it.

$$\begin{aligned}
\frac{d}{dx}(f(y)) &= \frac{d}{dx} g(x) \\
f'(y(x))y'(x) &= g'(x) \\
y'(x) &= \frac{g'(x)}{f'(y)}
\end{aligned}$$

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