The final exam is comprehensive. Make sure to review all subjects covered during class.

Area

Area has the following properties:

1. The area of a plane region is a non-negative real number.
2. The area of a rectangle is the product of its length and width.
3. Congruent regions have equal areas.
4. The area of the union of two regions that overlap only in a line segment is the sum of the areas of the two regions.
5. If one region is contained in a second region, then the area is less or equal to that of the second.

It is realitvely easy to calculate the area of any polygon: the canonical way is to devide it into triangles, which is always possible, and then add up the area of the triangles.

For regions, which are bounded by a curve (such as a circle), we approximate the area by inscribing and circumscribing polygons. By choosing smaller and smaller polygons, and letting their number go to infinity, we can calculate any reasonalbe area.

During this process, we use the $\sum$-summation:

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

Properties and special formulas:

1. **Linearity**
   (a) $\sum_{i=1}^{n} ca_i = c\sum_{i=1}^{n} a_i$
   (b) $\sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$
2. Collapsing sums: $\sum_{i=1}^{n}(a_{i+1} - a_i) = a_{n+1} - a_1$
3. $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$

The definite integral

The definite integral of a function f over a certain (closed) interval $[a, b]$ is calculated using the Riemann sum.

1. Choose a suitable partition $P : a = x_0 < x_1 < \ldots < x_n = b$.
2. Let $\Delta x_i = x_i - x_{i-1}$ be the length of the i$^{\text{th}}$ intervall.
3. On each subinterval $[x_{i-1}, x_i]$ pick an arbitrary point $\overline{x}_i$
4. The sum $R_P = \sum_{i_1}^{n} f(\overline{x}_i)\Delta x_i$ is called the **Riemann sum** for f on $[a, b]$ corresponding to the partition P.
5. Now take the limit as the intervals of the partition becomes smaller and smaller: let the largest subintervall (and with it of course all the others) of the partition go to zero.

$$\lim_{|P|\to 0} \sum_{i=1}^{n} f(\overline{x}_i)\Delta x_i.$$

6. If this limit exists we call it the definite integral of f on $[a, b]$ and denote it by

$$\int_a^b f(x)dx$$

We say that f is integrable.

There are more integrable functions, than differentiable functions. Functions which are continuous:

- Continuous functions on $[a, b]$.
- Bounded functions on $[a, b]$ which are continuous except at finitely many points.
- There are many more...

Properties:

1. **Linearity**:
   (a) $\int_a^b kf(x)dx = k\int_a^b f(x)dx$
   (b) $\int_a^b (f(x) \pm g(x))dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
2. **Additivity**: If f is integrable on an interval containing a, b and c in any ordre then
$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$
3. **Comparison**: If f and g are integrable on $[a, b]$ and $f(x) \leqslant g(x)$ then
$$\int_a^b f(x)dx \leqslant \int_a^b g(x)dx$$
4. **Boundedness**: If f is integrable on $[a, b]$ and $m \leqslant f(x) \leqslant M$ then
$$m(b-a) \leqslant \int_a^b f(x)dx \leqslant M(b-a)$$
5. Invert the boundaries:
$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$
and in particular we have
$$\int_a^a f(x)dx = 0$$
6. **Mean Value Theorem for Integrals**: If f is continuous on $[a, b]$, then there is a number c between a and b such that
$$f(c) = \frac{1}{b-a}\int_a^b f(t)dt$$
The right-hand side is the average value of the function over this specific interval.
7. **Symmetric functions**:
   (a) If f is even, then $\int_{-a}^a f(x)dx = 2\int_0^a f(x)dx$.
   (b) If f is odd, then $\int_{-a}^a f(x)dx = 0$.

8. **Periodic functions**: If f is periodic with period p, then

$$\int_{a+p}^{b+p} f(x)dx = \int_a^b f(x)dx$$

9. **Substitution for definite integrals** Let g have a continuous derivative on $[a,b]$ and f be continuous on the range of g. Then

$$\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

where $u = g(x)$.

10. **Area between the graph of two functions**: The area A between two functions f and g which are integrable on an interval $[a,b]$ is the difference between their integrals on this interval

$$A = \int_a^b f(x)dx - \int_a^b g(x)dx = \int_a^b (f(x) - g(x))\, dx$$

The indefinite integral

This is the inverse to differentiation, threfore also called anti-derivative. F is called an antiderivative for f if $F' = f$. We denote by $\int f(x)dx$ the family of **all** antoderivatives of f. They differ only by a constant.

In certain cases we know how to find an antiderivative.

1. **Power Rule**: For any rational number $r \neq -1$

$$\int x^r dx = \frac{x^{r+1}}{r+1} + C$$

so we can calculate the integral of any power of x including roots. An exception is $f(x) = \frac{1}{x}$.

2. **Certain trigonometric functions**:

$$\int \sin x dx = -\cos x + C$$
$$\int \cos x dx = \sin x + C$$

Pay attention to the sign!

3. **Generalised Power Rule**: If g is differentiable and r any rational number $r \neq -1$

$$\int [g(x)]^r g'(x)dx = \frac{[g(x)]^{r+1}}{r+1} + C$$

4. **Linearity**:

   (a) $\int kf(x)dx = k\int f(x)dx$

   (b) $\int(f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$

   Now we can calculate the integral of sums, differences and multiples of the above mentioned functions.

5. **Substitution**: Let g b differentiable and F an antiderivative of f. Then

$$\int f(g(x))g'(x)dx = F(g(x)) + C$$

Interaction between differntiation and integration

Differentiation and integration are both limit processes. They are closely related. Their relationship is expressed in the first and second fundamental theorem of calculus.

Theorem. *Let f be contiuous on $[a, b]$ and x a variable point on (a, b), then*

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Theorem. *Let f be continuous on $[a, b]$ and F an antiderivative of f, then*

$$\int_a^b f(x)dx = F(b) - F(a)$$

This is very useful to calculate integrals without having to calculate the Riemann sum.