Math 1210-009 Fall 2013

Final Examination

Monday, 16th December 2013, 18:00-20:00

Name: Solution

- No cell phones, computers, etc.
- No cheating.
- No notes, cheat sheets, books, etc.
- Write your name on each page.
- Show your work to get full credit.
- Make sure that what you write down is mathematically correct, e.g. don't forget equal signs etc.

	1	2	3	4	5	6	7	8	Extra Credit	$\sum$
Possible points	10	15	10	15	10	10	15	15	5	100+5
Your points										

1. **Evaluate the following integrals:**

(a)

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^4\theta \cos\theta d\theta$$

$u = \sin\theta$

$\frac{du}{d\theta} = \cos\theta$

$du = \cos\theta \, d\theta$

if $\theta = \frac{\pi}{2} \Rightarrow u = 1$

if $\theta = -\frac{\pi}{2} \Rightarrow u = -1$

$$\int_{-\pi/2}^{\pi/2} \sin^4\theta \cos\theta \, d\theta = \int_{-1}^{1} u^4 du = \left[\frac{u^5}{5}\right]_{-1}^{1} = \frac{1^5}{5} - \frac{(-1)^5}{5} = \frac{2}{5}$$

(b)

$$\int_2^2 \frac{\tan(3x)\cos(x)}{\sin 2x} dx$$

This equals to 0 as the upper and lower boundary coincide

2. Draw the graph of the function

$$f(x) = \frac{x^2}{(x-\frac{1}{2})^2}.$$

You may use that

$$f'(x) = \frac{-x}{(x-\frac{1}{2})^3}$$

$$f''(x) = \frac{2x+\frac{1}{2}}{(x-\frac{1}{2})^4}.$$

domain : $D_f = \mathbb{R} \setminus \{\frac{1}{2}\}$

zeros : $f(x) = 0 \iff x^2 = 0 \iff x = 0 \qquad P_0 = (0,0)$

y-intercept : $f(0) = 0 \Rightarrow P_0 = (0,0)$

critical points : - no endpoints

- $f'(x)$ DNE at $x = \frac{1}{2}$ but this is not part of the domain.
- $f'(x) = 0 \iff x = 0 \Rightarrow P_0 = (0,0)$

MAX/MIN : $f''(0) = \frac{\frac{1}{2}}{(-\frac{1}{2})^4} > 0 \Rightarrow P_0 = (0,0)$ is MIN

inflection points : - $f''(x)$ DNE at $x = \frac{1}{2}$ but this is not part of the domain

- $f''(x) = 0 \iff 2x + \frac{1}{2} = 0 \iff x = -\frac{1}{4}$

$f(-\frac{1}{4}) = \frac{\frac{1}{16}}{(-\frac{1}{4}-\frac{1}{2})^2} = \frac{\frac{1}{16}}{(-\frac{3}{4})^2} = \frac{1}{9} \Rightarrow P_1 = (-\frac{1}{4}, \frac{1}{9})$

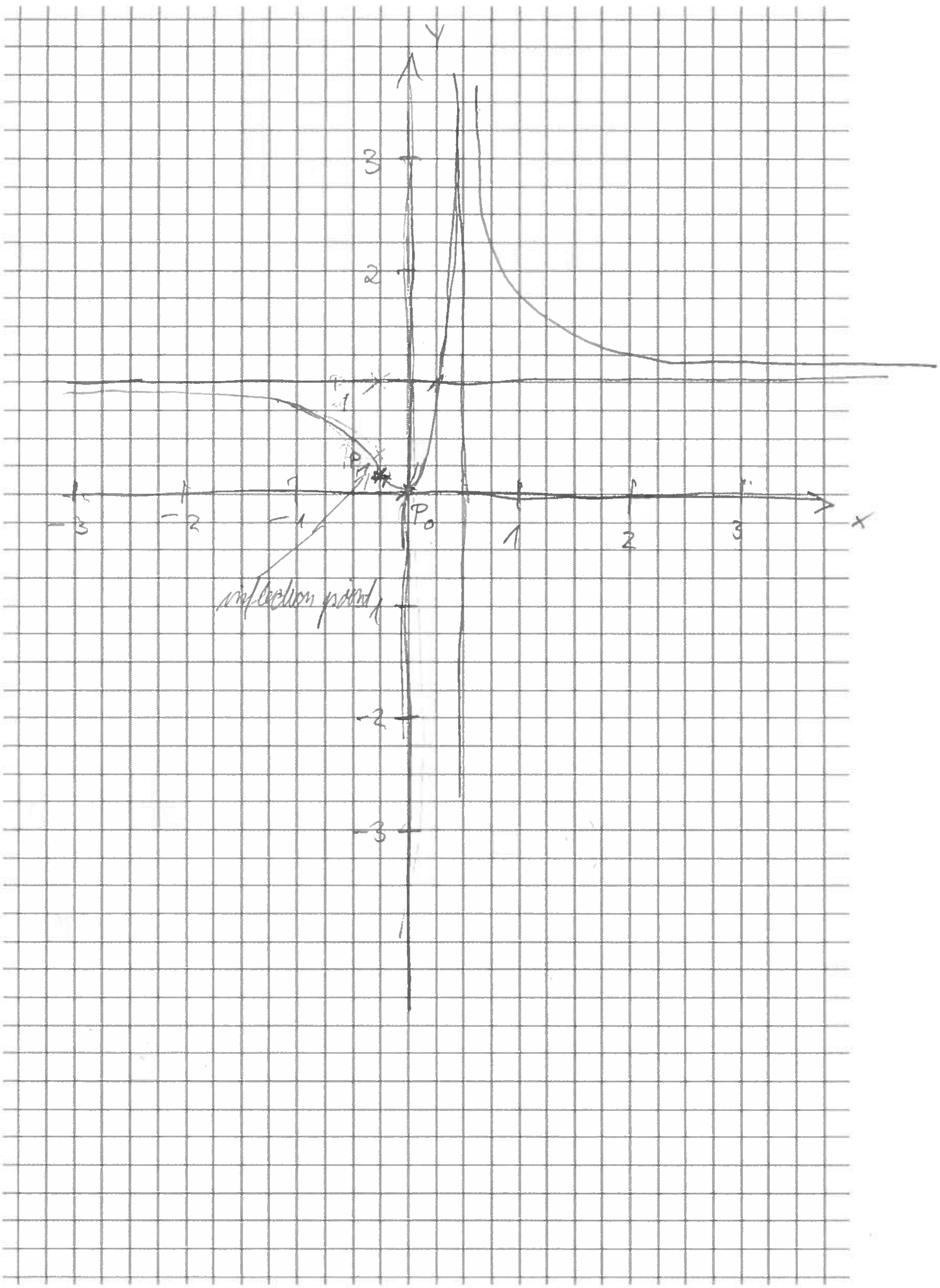
$\left.\begin{array}{ll} f''(x) > 0 & \text{for } x > -\frac{1}{4} \\ f''(x) < 0 & \text{for } x < -\frac{1}{4} \end{array}\right\} \Rightarrow P_1$ is inflection point.

assymptotes : vertical : $\lim_{x\to\frac{1}{2}^+} \frac{x^2}{(x-\frac{1}{2})^2} = +\infty \qquad \lim_{x\to\frac{1}{2}^-} \frac{x^2}{(x-\frac{1}{2})^2} = +\infty$

because squares are always positive in $\mathbb{R}$.

horizontal : $\lim_{x\to+\infty} \frac{x^2}{(x-\frac{1}{2})^2} = \lim_{x\to+\infty} \frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2} - \frac{x}{x^2} + \frac{1}{4x^2}} = \frac{1}{1-0+0} = 1$

$\lim_{x\to-\infty} \frac{x^2}{(x-\frac{1}{2})^2} = \lim_{x\to-\infty} \frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2} - \frac{x}{x^2} + \frac{1}{4x^2}} = \frac{1}{1-0+0} = 1$


y
3
2
1
P_1
P_0
-3
-2
-1
1
2
3
x
inflection point
-2
-3

3. In each of the following problems find the indicated limit or state that it does not exist.

(a)

$$\lim_{x\to 0}\frac{\tan(3x)}{\sin 2x}$$

Recall that $\tan x = \frac{\sin x}{\cos x}$.

$$\lim_{x\to 0}\frac{\tan(3x)}{\sin(2x)} = \lim_{x\to 0}\frac{\sin 3x}{\cos 3x \cdot \sin 2x} =$$

$$= \lim_{x\to 0}\left(\frac{1}{\cos 3x}\right)\cdot\left(3\cdot\frac{\sin 3x}{3x}\right)\cdot\left(\frac{2x}{2\sin 2x}\right) =$$

$$= 1\cdot(3\cdot 1)\cdot\frac{1}{2}\cdot 1 = \frac{3}{2}$$

(b) The right-hand limit:

$$\lim_{x\to 1^+}\frac{|x^4-1|}{x^2-1}$$

For $x>1$ the term $x^4-1>0$ $\Rightarrow$ we don't need the absolut value for the right-handed limit.

$$\lim_{x\to 1^+}\frac{|x^4-1|}{x^2-1} = \lim_{x\to 1^+}\frac{x^4-1}{x^2-1} = \lim_{x\to 1^+}\frac{(x^2-1)(x^2+1)}{x^2-1} =$$

$$= \lim_{x\to 1^+}(x^2+1) = 2$$

4. Find the equation of the tangent line to the curve

$$f(x) = 2\sqrt{x+4} + 5x^2 + 3x$$

at $x = 0$.

The tangent line of f at $x=0$ has the same slope as f at this point.
It is given by the first derivative:

$$f'(x) = 2 \cdot \frac{1}{2\sqrt{x+4}} + 5 \cdot 2x + 3 = \frac{1}{\sqrt{x+4}} + 10x + 3$$

$$m = f'(0) = \frac{1}{\sqrt{0+4}} + 10 \cdot 0 + 3 = \frac{1}{2} + 3 = 3.5$$

The tangent line of f at $x=0$ passes through the point $(0, f(0))$:

$$f(0) = 2\sqrt{0+4} + 5 \cdot 0^2 + 3 \cdot 0 = 2\sqrt{4} = 4$$

The equation for the tangent line has the form

$$y = l(x) = mx + t$$

With $m = 3.5$: $y = 3.5x + t$

With $P = (0, 4)$: $4 = 3.5 \cdot 0 + t = t \Rightarrow t = 4$

$$\Rightarrow y = l(x) = 3.5x + 4$$

5. (a) State the First Fundamental Theorem of Calculus.

Let f be a continuous function on $[a,b]$ and x a variable in $[a,b]$.

Then

$$\frac{d}{dx}\int_a^x f(z)\,dz = f(x)$$

(b) Find $G'(x)$ given $G(x) = \int_1^x \sqrt{2t + \sin t}\,dt$. Justify your unswer.

Let $f(z) = \sqrt{2z + \sin z}$: This is well defined for $z \geq 1$, and continuous.

Therefore we can apply the first fundamental theorem of calculus.

$$G'(x) = \frac{d}{dx}\int_1^x \sqrt{2z + \sin z}\,dz = \sqrt{2x + \sin x}$$

6. Find the first derivative of the following functions.

(a)

$$f(x) = \frac{x+1}{\sqrt{x-1}}$$

$$f'(x) = \frac{\sqrt{x-1}\cdot 1 - (x+1)\frac{1}{2\sqrt{x-1}}}{\sqrt{x-1}^2} =$$

$$= \frac{\sqrt{x-1} - \frac{x+1}{2\sqrt{x-1}}}{(x-1)^2} =$$

$$= \frac{2\sqrt{x-1}^2 - (x+1)}{2\sqrt{x-1}(x-1)^2} =$$

$$= \frac{2(x-1) - (x+1)}{2\sqrt{x-1}^3} =$$

$$= \frac{x-3}{2\sqrt{x-1}^3}$$

(b)

$$f(x) = \tan\frac{1}{x}$$

Calculate the first derivative of $\tan x$:

$$\tan(x)' = \left(\frac{\sin x}{\cos x}\right)' =$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} =$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$f'(x) = \frac{1}{\cos^2(\frac{1}{x})} \cdot \left(-\frac{1}{x^2}\right) = -\frac{1}{x^2 \cos^2 \frac{1}{x}}$$

7. (a) Give the definition of the derivative of a function.

Let f be differentiable.

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

(b) Use this definition to find the derivative of the function $f(x) = x^2 + 4x - 7$.

$$f'(x) = \lim_{h\to 0} \frac{\left((x+h)^2 + 4(x+h) - 7\right) - (x^2+4x-7)}{h} =$$

$$= \lim_{h\to 0} \frac{(x^2 + 2hx + h^2 + 4x + 4h - 7 - x^2 - 4x - 7)}{h} =$$

$$= \lim_{h\to 0} \frac{2hx + h^2 + 4h}{h} =$$

$$= \lim_{h\to 0} (2x + h + 4) = 2x + 4$$

(c) Use this definition to find the derivative of the function $f(x) = \frac{2}{x+3}$.

$$f'(x) = \lim_{h\to 0} \frac{\frac{2}{(x+h)+3} - \frac{2}{x+3}}{h} =$$

$$= \lim_{h\to 0} \frac{(x+3)2 - 2(x+h+3)}{(x+h+3)(x+3)\cdot h} =$$

$$= \lim_{h\to 0} \frac{2x + 6 - 2x - 2h - 6}{(x+h+3)(x+3)\cdot h} =$$

$$= \lim_{h\to 0} \frac{-2h}{(x+h+3)(x+3)\cdot h} =$$

$$= \lim_{h\to 0} -\frac{2}{(x+h+3)(x+3)} = -\frac{2}{(x+3)^2}$$

8. Let $f(x) = 2x + 1$ on the interval $[0, 1]$.

(a) Calculate the Riemann sum for f for the partition

$$P : x_0 = 0 < x_1 = \frac{1}{4} < x_2 = \frac{1}{2} < x_3 = \frac{3}{4} < x_4 = 1$$

$\forall i \in \{1, \dots, 4\}: \Delta x_i = \frac{1}{4}$

choose $\bar{x}_i = x_{i-1} \Rightarrow \bar{x}_1 = 0, \bar{x}_2 = \frac{1}{4}, \bar{x}_3 = \frac{1}{2}, \bar{x}_4 = \frac{3}{4}$.

$f(\bar{x}_1) = 1, f(\bar{x}_2) = 1.5, f(\bar{x}_3) = 2 \quad f(\bar{x}_4) = 2.5$

$$R_P = \sum_{i=1}^{4} f(\bar{x}_i) \cdot \Delta x_i$$

$$= 1 \cdot \frac{1}{4} + 1.5 \cdot \frac{1}{4} + 2 \cdot \frac{1}{4} + 2.5 \cdot \frac{1}{4} =$$

$$= \frac{1}{4} + \frac{3}{8} + \frac{1}{2} + \frac{5}{8} = \frac{14}{8}$$

(b) Calculate the Riemann sum for f for the partition in n equal sub-intervals

$$P : x_0 = 0 < x_1 = \frac{1}{n} < x_2 = \frac{2}{n} < x_3 = \frac{3}{n} < \cdots < x_{n-1} = \frac{n-1}{n} < x_n = 1$$

$\forall i \in \{1, \dots, n\}: \Delta x_i = \frac{1}{n}$

Choose $\bar{x}_i = x_{i-1} \Rightarrow \bar{x}_1 = 0, \bar{x}_2 = \frac{1}{n}, \bar{x}_3 = \frac{2}{n}, \dots, \bar{x}_i = \frac{i-1}{n}, \dots, \bar{x}_n = \frac{n-1}{n}$.

$f(\bar{x}_1) = 1, f(\bar{x}_2) = \frac{2+n}{n}, f(\bar{x}_3) = \frac{4+n}{n}, \dots, f(\bar{x}_i) = \frac{2(i-1)+n}{n}, \dots, f(\bar{x}_n) = \frac{2(n-1)+n}{n}$

$$R_P = \sum_{i=1}^{n} f(\bar{x}_i) \cdot \Delta x_i = \sum_{i=1}^{n} \frac{2(i-1)+n}{n} \cdot \frac{1}{n} =$$

$$= \frac{2}{n^2} \sum_{i=1}^{n} (i-1) + \frac{1}{n} \sum_{i=1}^{n} 1 =$$

$$= \frac{2}{n^2} \sum_{j=1}^{n-1} j + \frac{1}{n} \cdot n =$$

$$= \frac{2}{n^2} \left(\frac{n \cdot (n-1)}{2} \right) + 1 = \frac{n-1}{n} + 1$$

(c) Use the previous results to calculate the integral

$$\int_0^1 (2x+1)dx.$$

$$\int_0^1 (2x+1)\,dx = \lim_{|P|\to 0} R_P =$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} f(\bar{x}_i)\Delta x_i =$$

$$= \lim_{n\to\infty} \frac{n-1}{n} + 1 =$$

$$= \lim_{n\to\infty} \left(1 - \frac{1}{n} + 1\right) = 2$$

Extra Credit. Consider the curve given by the function $y = 0.1x^2$ and the point $Q = (0;2)$. Which are the point(s) P on the curve that **minimise** the distance to the point Q?
The distance between two points $A = (a_1, a_2)$ and $B = (b_1, b_2)$ is given by

$$d = \sqrt{(a_1 - b_1)^2 + (a_2 - b_2)^2}$$

A point P on the curve is given by $(x, 0.1x^2)$.

$$P - Q = (x - 0, 0.1x^2 - 2) = (x, 0.1x^2 - 2)$$

$$d(x) = \sqrt{x^2 + (0.1x^2 - 2)^2}$$

$$d'(x) = \frac{1}{2\sqrt{x^2 + (0.1x^2-2)^2}} \cdot \left(2x + 2(0.1x^2 - 2) \cdot 0.2x\right)$$

$$= \frac{2x + 0.04x^3 - 0.8x}{2\sqrt{x^2 + (0.1x^2-2)^2}}$$

$$= \frac{1.2x + 0.04x^3}{2\sqrt{x^2 + (0.1x^2-2)^2}}$$

$d'(x) = 0$ if $0 = 1.2x + 0.04x^3 = x(1.2 + 0.04x^2)$

The only solution in $\mathbb{R}$ is $x = 0$

This is really a minimum as $d'(x) < 0$ for $x < 0$
$d'(x) > 0$ for $x > 0$

$\Rightarrow d(x)$ has a minimum at $x = 0$

$P = (x, f(x)) \Rightarrow P_0 = (0, f(0)) = (0, 0.1 \cdot 0^2) = (0,0)$