Math 1210-009 Fall 2013

Second Midterm Examination

28th October 2013

Name:

- No cell phones, computers, etc.
- No cheating.
- No notes, cheat sheets, books, etc.
- Write your name on each page.
- Show your work to get full credit.
- Make sure that what you write down is mathematically correct, e.g. don't forget equal signs etc.

	1	2	3	4	$\sum$	%
Possible points	15	15	10	10	50	100
Your points						

Avarage: 80,5 %

1. First order derivatives.

   (a) Show that the first derivative of the following function is $\frac{2}{x^2}$. (5 points)

$$f(x) = \frac{x^2 - 4x + 4}{x(x-2)}$$

$$f(x) = \frac{(x-2)^2}{x(x-2)} = \frac{x-2}{x}$$

$$f'(x) = \frac{x - (x-2)}{x^2} = \frac{2}{x^2}$$

or

$$f'(x) = \frac{x(x-2)(2x-4) - (x^2-4x+4)(x+(x-2))}{x^2(x-2)^2} =$$

$$= \frac{x \cdot 2(x-2)^2 - (x-2)^2(2x-2)}{x^2(x-2)^2} =$$

$$= \frac{2x - 2x + 2}{x^2} = \frac{2}{x^2}$$

   (b) FInd the first derivative of the following function. (5 points)

$$f(x) = 1 - \cos^2 x$$

$$f(x) = \sin^2 x$$

$$f'(x) = 2\sin x \cdot \cos x$$

or

$$f'(x) = 0 - 2\cos x \cdot (-\sin x) = 2\cos x \sin x$$

(c) Show that the first derivative of the following function is $\frac{4x}{\sqrt{2-x^2}^3}$. (5points)

$$f(x) = \frac{4}{\sqrt{2-x^2}}$$

$$f'(x) = \frac{-4\left(\frac{1}{2\sqrt{2-x^2}}\right)\cdot(-2x}{(2-x^2)} =$$

$$= \frac{4x}{(2-x^2)\sqrt{2-x^2}} =$$

$$= \frac{4x}{\sqrt{2-x^2}^3}$$

or

$$f(x) = 4\cdot(2-x^2)^{-1/2}$$

$$f'(x) = 4\left(-\frac{1}{2}\right)\cdot(2-x^2)^{-3/2}(-2x) =$$

$$= 4x(2-x^2)^{-3/2} =$$

$$= \frac{4x}{\sqrt{2-x^2}^3}$$

2. Show that the third order derivative of the following function is $\frac{\cancel{8}\,18}{(1-x)^4}$. (15 points)

$$f(x) = \frac{3x}{1-x}$$

$$\frac{df}{dx}(x) = f'(x) = \frac{(1-x)3 - 3x(-1)}{(1-x)^2} = \frac{3 - 3x + 3x}{(1-x)^2} = \frac{3}{(1-x)^2}$$

$$\frac{d^2f}{dx}(x) = f''(x) = \frac{-3\cdot 2(1-x)(-1)}{(1-x)^4} = \frac{6}{(1-x)^3}$$

$$\frac{d^3f}{dx}(x) = f^{(3)}(x) = \frac{-6\cdot 3(1-x)^2(-1)}{(1-x)^6} = \frac{18}{(1-x)^4}$$

3. Recall that $n! = n \cdot (n-1) \cdot (n-2) \cdots 3 \cdot 2 \cdot 1$. The goal of this problem is to find a formula for

$$\frac{d^n}{dx^n}\left(\frac{1}{(-1)^n \cdot (n!) \cdot x}\right).$$

(a) Calculate the first five derivatives of the function $f(x) = \frac{1}{x}$. (5 points)

$$\frac{df}{dx} = -\frac{1}{x^2}$$

$$\frac{d^2f}{dx^2} = \frac{2}{x^3}$$

$$\frac{d^3f}{dx^3} = -\frac{6}{x^4}$$

$$\frac{d^4f}{dx^4} = \frac{24}{x^5}$$

$$\frac{d^5f}{dx^5} = -\frac{120}{x^6}$$

(b) Describe the pattern that you discover in the previous part and use this to give a formula for the following (~~2~~ 5 points)

$$\frac{d^n}{dx^n}\left(\frac{1}{x}\right).$$

- alternating signs: -1 for n odd, $+1$ for n even

- numerator : $n!$

- denominator x^{n+1}

$$\Rightarrow \quad \frac{d^n}{dx^n}\left(\frac{1}{x}\right) = (-1)^n \frac{n!}{x^{n+1}}$$

(c) Use the previous result to give a formula for the following (2 points) for extra credit

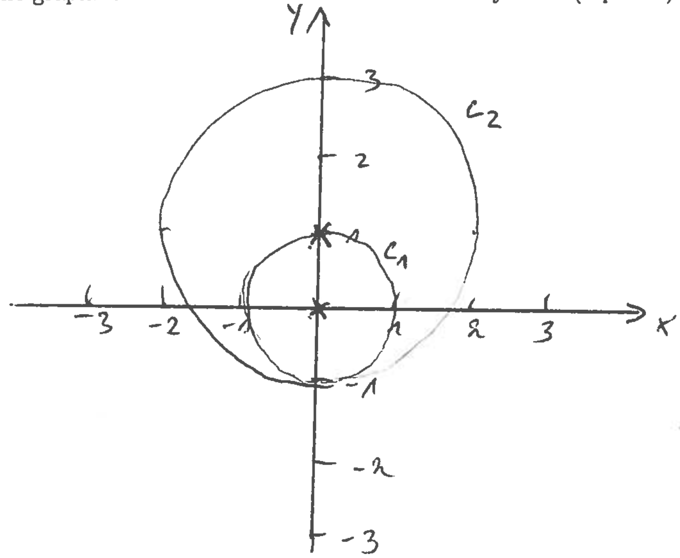
$$\frac{d^n}{dx^n}\left(\frac{1}{(-1)^n \cdot (n!) \cdot x}\right).$$

$$\frac{d^n}{dx^n}\left(\frac{1}{(-1)^n \cdot (n!) \cdot x}\right) = \frac{1}{(-1)^n \cdot n!} \cdot \frac{d^n}{dx^n}\left(\frac{1}{x}\right) = \frac{(-1)^n}{n!}(-1)^n \frac{n!}{x^{n+1}} = \frac{1}{x^{n+1}}$$

4. Consider the two following circles:

$$c_1 \; : \; x^2 + y^2 = 1 \quad \text{centre } (0,0) \text{ and radius } 1$$
$$c_2 \; : \; x^2 + (y-1)^2 = 4 \quad \text{centre } (0,1) \text{ and radius } 2$$

(a) Sketch the graphs of these two circles in a coordinate system. (2 points)



(b) Calculate their intersection point P. (3 points)

$c_1: \; x^2 = 1 - y^2$

$c_2: \; x^2 = 4 - (y-1)^2$

$c_1 = c_2: \quad 1 - y^2 = 4 - (y-1)^2$

$1 - y^2 = 4 - (y^2 - 2y + 1)$

$1 - y^2 = 3 - y^2 + 2y$

$-2 = 2y$

$\underline{\underline{-1 = y}}$

$y = -1$ in c_1: $\quad x^2 = 1 - (-1)^2 = 0 \quad \Rightarrow x = 0$

$P = (0, -1)$

(c) Use implicit differentiation to find the slopes of the circles at this intersection point. (4 points)

$\frac{d}{dx} C_1$: $2x + 2y\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x}{y}$

$\frac{dy}{dx}(0,-1) = -\frac{0}{(-1)} = 0$

$\frac{d}{dx} C_2$: $2x + 2(y-1)\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x}{(y-1)}$

$\frac{dy}{dx}(0,-1) = -\frac{0}{(-1-1)} = 0$

(d) At what angle do the tangent lines of the cirlces at the point P intersect? Mark the correct answer(s): (1 point)

- At a right angle.
- Both circles have the same tangent lines at this point.
- At a 0-degree angle.
- Between 45 and 90 degree.